COUPLED PAINLEVÉ VI SYSTEMS IN DIMENSION FOUR WITH AFFINE WEYL GROUP SYMMETRY OF TYPES $B_6^{(1)}$, $D_6^{(1)}$ AND $D_7^{(2)}$

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ABSTRACT. We find four kinds of six-parameter family of coupled Painlevé VI systems in dimension four with affine Weyl group symmetry of types $B_6^{(1)}$, $D_6^{(1)}$ and $D_7^{(2)}$. Each system is the first example which gave higher-order Painlevé equations of types $B_l^{(1)}$, $D_l^{(1)}$ and $D_l^{(2)}$, respectively. Each system can be expressed as a polynomial Hamiltonian system. We show that these systems are equivalent by an explicit birational and symplectic transformation, respectively. By giving each holomorphy condition, we can recover each system. These symmetries, holomorphy conditions and invariant divisors are new. We also give an explicit description of a confluence process from the system of type $D_6^{(1)}$ to the system of type $A_5^{(1)}$ by taking the coupling confluence process from the Painlevé VI system to the Painlevé V system.

1. INTRODUCTION

In 1912, considering the significant problem of searching for higher-order analogues of the Painlevé equations, Garnier discovered a series of systems of nonlinear partial differential equations, which can be considered as a generalization of the Painlevé VI equation from the viewpoint of monodromy preserving deformations of the second-order linear ordinary differential equations, now called the Garnier system (see [12]).

From the viewpoint of affine Weyl groups, a series of systems of nonlinear ordinary differential equations with affine Weyl group symmetry of type $A_l^{(1)}$ were studied (cf. [10]). This series gives a generalization of Painlevé equations P_{IV} and P_V to higher orders.

The Painlevé VI equation has symmetry under the affine Weyl group of type $D_4^{(1)}$. On the other hand, the generalizations of the systems of type $A_l^{(1)}$ do not include the Painlevé VI equation. Thus, it is an important remaining problem to find a generalization of the Painlevé VI equation for which the symmetries can be established. In the present paper, we find a 6-parameter family of coupled Painlevé VI systems in dimension four with affine Weyl group symmetry of type $D_6^{(1)}$. Our differential system is equivalent to a Hamiltonian system given by

$$(1)\qquad \frac{dx}{dt}=\frac{\partial H}{\partial y},\quad \frac{dy}{dt}=-\frac{\partial H}{\partial x},\quad \frac{dz}{dt}=\frac{\partial H}{\partial w},\quad \frac{dw}{dt}=-\frac{\partial H}{\partial z}$$

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with the polynomial Hamiltonian

$$\begin{aligned} H = &H_{VI}(x,y,t;\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)+H_{VI}(z,w,t;\beta_0,\beta_1,\beta_2,\beta_3,\beta_4)\\ &+\frac{2(x-t)yz\{(z-1)w+\beta_2\}}{t(t-1)}. \end{aligned} \tag{2}$$

Here x, y, z, w denote unknown complex variables, and $\alpha_0, \alpha_1, \alpha_2, \alpha_4, \beta_2, \beta_3, \beta_4$ are complex parameters satisfying the relation:

$$\alpha_0+\alpha_1+2\alpha_2+2(\alpha_4-\beta_4)+2\beta_2+\beta_3+\beta_4=1. \tag{3}$$

We remark that for this system we tried to seek its first integrals of polynomial type with respect to x, y, z, w. However, we can not find. Of course, the Hamiltonian H is not its first integral.

Remark 1.1. The parameters $\alpha_0, \alpha_1, \dots, \alpha_4, \beta_0, \beta_1, \dots, \beta_4$ satisfy the relations:

$$\alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_4=1, \ \beta_0+\beta_1+2\beta_2+\beta_3+\beta_4=1, \tag{4}$$

$$\alpha_1+2\alpha_2+\alpha_4-\beta_1-\beta_4=0, \ \alpha_3-\alpha_4-2\beta_2-\beta_3+\beta_4=0. \tag{5}$$

From these relations, it is easy to see that the parameters $\alpha_3, \alpha_4, \beta_0, \beta_1$ are described as linear combinations of the basis elements $\alpha_0, \alpha_1, \alpha_2, \beta_2, \beta_3, \beta_4$, which is explicitly given as follows:

$$\begin{aligned} &\alpha_3=\frac{1-\alpha_0-\alpha_1-2\alpha_2+2\beta_2+\beta_3-\beta_4}{2}, \ \alpha_4=\frac{1-\alpha_0-\alpha_1-2\alpha_2-2\beta_2-\beta_3+\beta_4}{2},\\ &\beta_0=\frac{1+\alpha_0-\alpha_1-2\alpha_2-2\beta_2-\beta_3-\beta_4}{2}, \ \beta_1=\frac{1-\alpha_0+\alpha_1+2\alpha_2-2\beta_2-\beta_3-\beta_4}{2}. \end{aligned} \tag{6}$$

The relations (4) are well-known as the parameter's relation of the Painlevé VI system, and the relations (5) are new. This representation of type $D_6^{(1)}$ can be constructed by coupling two copies of the $D_4^{(1)}$ root system by the relations (4) and (5) (see Theorem 2.3). This representation is new.

The symbol $H_{VI}(q,p,t;\delta_0,\delta_1,\delta_2,\delta_3,\delta_4)$ denotes the Hamiltonian of the second-order Painlevé VI equation given by

$$\begin{aligned} &H_{VI}(q,p,t;\delta_0,\delta_1,\delta_2,\delta_3,\delta_4)\\ &=\frac{1}{t(t-1)}[p^2(q-t)(q-1)q-\{(\delta_0-1)(q-1)q+\delta_3(q-t)q\\ &+\delta_4(q-t)(q-1)\}p+\delta_2(\delta_1+\delta_2)q] \quad (\delta_0+\delta_1+2\delta_2+\delta_3+\delta_4=1). \end{aligned} \tag{7}$$

This system is the first example which gave higher-order Painlevé equations of type $D_l^{(1)}$.

We also give an explicit description of a confluence process to the system with affine Weyl group symmetry of type $A_5^{(1)}$ (cf. [10]). Additionally our results here, we obtain a new approach to the study of various higher-order Painlevé equations, presented in a

series of papers for which this is the first. These papers are aimed at a complete study of the following problem:

PROBLEM 1.2. For each affine root system A with affine Weyl group $W(A)$, find a system of differential equations for which $W(A)$ acts as its Bäcklund transformations.

We remark that the Bäcklund transformations of the system of type $D_6^{(1)}$ satisfy

$$s_i(g) = g + \frac{\alpha_i}{f_i}\{f_i, g\} + \frac{1}{2!}\left(\frac{\alpha_i}{f_i}\right)^2 \{f_i, \{f_i, g\}\} + \cdots \quad (g \in \mathbb{C}(t)[x, y, z, w]), \tag{8}$$

where poisson bracket $\{,\}$ satisfies the relations:

$$\{y, x\} = \{w, z\} = 1, \quad \textit{the others are } 0. \tag{9}$$

Since these Bäcklund transformations have Lie theoretic origin, similarity reduction of a Drinfeld-Sokolov hierarchy admits such a Bäcklund symmetry.

This paper is organized as follows. In Section 2, we state our motivation and main results. In Section 3, we present two types of a 6-parameter family of coupled Painlevé VI systems in dimension four with affine Weyl group symmetry of type $B_6^{(1)}$. In Section 4, we find a 6-parameter family of coupled Painlevé VI systems in dimension four with affine Weyl group symmetry of type $D_7^{(2)}$. In Section 5, we find an autonomous version of the system of type $D_6^{(1)}$. In Section 6, we will give a brief review on the systems of types $A_5^{(1)}$ and $A_4^{(1)}$. In Section 7, we will explain our approach for obtaining the system (1). In Section 8, we will prove Theorems 2.11 and 2.12.

2. Motivation and main results

In the works [16, 17, 18], the author studied higher-order Painlevé equations from a viewpoint of polynomial Hamiltonian systems. In the case of the second-order Painlevé equations, let us summarize the following important properties of the Painlevé equations.

Notation.
- $H \in \mathbb{C}(t)[x, y]$,   • $deg(H)$: degree with respect to x, y.

symmetry	$W(D_4^{(1)})$	$W(A_3^{(1)})$	$W(A_2^{(1)})$	$W(C_2^{(1)})$	$W(A_1^{(1)})$
Painlevé equations	P_{VI}	P_V	P_{IV}	P_{III}	P_{II}
degree of Hamiltonian H	5	4	3	4	3

We are interested in polynomial Hamiltonian systems and symmetry under the affine Weyl group, and wish to search for higher-order Painlevé systems with these favorable properties. As examples of higher-order Painlevé systems, Adler (see [11, 20]) studied ordinary differential systems with affine Weyl group symmetry of type $A_l^{(1)}$. When $l = 2$ (resp. 3), this system of type $A_2^{(1)}$ (resp. $A_3^{(1)}$) is equivalent to the fourth (resp. fifth) Painlevé equation P_{IV} (resp. P_V). They are considered to be higher-order versions of P_V (resp. P_{IV}) when l is odd (resp. even). These two examples motivated the author to

find examples of higher-order versions other than P_V and P_{IV}. At first, we study four-dimensional case. Let us summarize important properties of the system of types $A_5^{(1)}$ and $A_4^{(1)}$.

symmetry	$W(A_5^{(1)})$	$W(A_4^{(1)})$
Hamiltonian H	$H_V(x,y,t)+H_V(z,w,t)$ $-2yzw+\frac{2xyzw}{t}$	$H_{IV}(x,y,t)+H_{IV}(z,w,t)$ $+2yzw$
differential system	coupled Painlevé V	coupled Painlevé IV
degree of Hamiltonian H	4	3

These properties suggest the possibility that there exists a procedure for searching for such higher-order versions with symmetry under the affine Weyl group of type $D_6^{(1)}$. Here, let us consider the following problem.

PROBLEM 2.1. Can we show existence of a coupled Painlevé VI system in dimension four satisfying the following assumptions $(A1),(A2)$? If yes, can we find it explicitly and is it unique?

ASSUMPTION 2.2. $(A1)$ $deg(H)=5$ with respect to x,y,z,w.
$(A2)$ The system has symmetry under the affine Weyl group of type $D_6^{(1)}$.

To answer this, in this paper, we present a 6-parameter family of coupled Painlevé VI systems in dimension four with extended affine Weyl group symmetry of type $D_6^{(1)}$ explicitly given by

$$\begin{cases} \dfrac{dx}{dt}=\dfrac{\partial H}{\partial y}=\dfrac{1}{t(t-1)}\{2y(x-t)(x-1)x-(\alpha_0-1)(x-1)x-\alpha_3(x-t)x \\ \qquad\qquad -\alpha_4(x-t)(x-1)+2(x-t)z((z-1)w+\beta_2)\}, \\ \dfrac{dy}{dt}=-\dfrac{\partial H}{\partial x}=\dfrac{1}{t(t-1)}[-\{(x-t)(x-1)+(x-t)x+(x-1)x\}y^2+\{(\alpha_0-1)(2x-1) \\ \qquad\qquad +\alpha_3(2x-t)+\alpha_4(2x-t-1)\}y-\alpha_2(\alpha_1+\alpha_2)-2yz((z-1)w+\beta_2)], \\ \dfrac{dz}{dt}=\dfrac{\partial H}{\partial w}=\dfrac{1}{t(t-1)}\{2w(z-t)(z-1)z-(\beta_0-1)(z-1)z-\beta_3(z-t)z \\ \qquad\qquad -\beta_4(z-t)(z-1)+2(x-t)yz(z-1)\}, \\ \dfrac{dw}{dt}=-\dfrac{\partial H}{\partial z}=\dfrac{1}{t(t-1)}[-\{(z-t)(z-1)+(z-t)z+(z-1)z\}w^2+\{(\beta_0-1)(2z-1) \\ \qquad +\beta_3(2z-t)+\beta_4(2z-t-1)\}w-\beta_2(\beta_1+\beta_2)-2(x-t)y((2z-1)w+\beta_2)] \end{cases}$$

with the polynomial Hamiltonian H (2).

THEOREM 2.3. *The system* (1) *admits extended affine Weyl group symmetry of type* $D_6^{(1)}$ *as the group of its Bäcklund transformations, whose generators* s_i $(i=0,\dots,6),\pi_j$ $(j=1,\dots,4)$ *are explicitly given as follows*: *with the notations*

$$\gamma_1:=\alpha_4-\beta_4, \qquad (*):=(x,y,z,w,t;\alpha_0,\alpha_1,\alpha_2,\gamma_1,\beta_2,\beta_3,\beta_4),$$

$$
\begin{aligned}
&s_0 : (*) \rightarrow \left(x, y - \frac{\alpha_0}{x-t}, z, w, t; -\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \gamma_1, \beta_2, \beta_3, \beta_4\right),\\
&s_1 : (*) \rightarrow (x, y, z, w, t; \alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \gamma_1, \beta_2, \beta_3, \beta_4),\\
&s_2 : (*) \rightarrow \left(x + \frac{\alpha_2}{y}, y, z, w, t; \alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \gamma_1 + \alpha_2, \beta_2, \beta_3, \beta_4\right),\\
&s_3 : (*) \rightarrow \left(x, y - \frac{\gamma_1}{x-z}, z, w + \frac{\gamma_1}{x-z}, t; \alpha_0, \alpha_1, \alpha_2 + \gamma_1, -\gamma_1, \beta_2 + \gamma_1, \beta_3, \beta_4\right),\\
&s_4 : (*) \rightarrow \left(x, y, z + \frac{\beta_2}{w}, w, t; \alpha_0, \alpha_1, \alpha_2, \gamma_1 + \beta_2, -\beta_2, \beta_3 + \beta_2, \beta_4 + \beta_2\right),\\
&s_5 : (*) \rightarrow \left(x, y, z, w - \frac{\beta_3}{z-1}, t; \alpha_0, \alpha_1, \alpha_2, \gamma_1, \beta_2 + \beta_3, -\beta_3, \beta_4\right),\\
&s_6 : (*) \rightarrow \left(x, y, z, w - \frac{\beta_4}{z}, t; \alpha_0, \alpha_1, \alpha_2, \gamma_1, \beta_2 + \beta_4, \beta_3, -\beta_4\right),\\
&\pi_1 : (*) \rightarrow (\frac{t(t-1)+t(x-t)}{x-t}, -\frac{(x-t)((x-t)y+\alpha_2)}{t(t-1)}, \frac{t(t-1)+t(z-t)}{z-t},\\
&\qquad -\frac{(z-t)((z-t)w+\beta_2)}{t(t-1)}, t; \alpha_1, \alpha_0, \alpha_2, \gamma_1, \beta_2, \beta_4, \beta_3),\\
&\pi_2 : (*) \rightarrow \left(\frac{t}{z}, -\frac{z(zw+\beta_2)}{t}, \frac{t}{x}, -\frac{x(xy+\alpha_2)}{t}, t; \beta_3, \beta_4, \beta_2, \gamma_1, \alpha_2, \alpha_0, \alpha_1\right),\\
&\pi_3 : (*) \rightarrow (1-x, -y, 1-z, -w, 1-t; \alpha_0, \alpha_1, \alpha_2, \gamma_1, \beta_2, \beta_4, \beta_3),\\
&\pi_4 : (*) \rightarrow (\frac{(t-1)x}{t-x}, \frac{(t-x)(ty-xy-\alpha_2)}{t(t-1)}, \frac{(t-1)z}{t-z}, \frac{(t-z)(tw-zw-\beta_2)}{t(t-1)},\\
&\qquad 1-t; \alpha_1, \alpha_0, \alpha_2, \gamma_1, \beta_2, \beta_3, \beta_4).
\end{aligned} \tag{10}
$$

We note that these transformations s_i, π_j are birational and symplectic.

The Bäcklund transformations of the system of type $D_6^{(1)}$ satisfy

$$
s_i(g) = g + \frac{\alpha_i}{f_i}\{f_i, g\} + \frac{1}{2!}\left(\frac{\alpha_i}{f_i}\right)^2 \{f_i, \{f_i, g\}\} + \cdots \quad (g \in \mathbb{C}(t)[x, y, z, w]), \tag{11}
$$

where poisson bracket $\{,\}$ satisfies the relations:

$$
\{y, x\} = \{w, z\} = 1, \quad \textit{the others are } 0. \tag{12}
$$

Since these Bäcklund transformations have Lie theoretic origin, similarity reduction of a Drinfeld-Sokolov hierarchy admits such a Bäcklund symmetry.

Proposition 2.4. *The system* (1) *has the following invariant divisors*:

parameter's relation	f_i
$\alpha_0=0$	$f_0:=x-t$
$\alpha_1=0$	$f_1:=x-\infty$
$\alpha_2=0$	$f_2:=y$
$\alpha_4=\beta_4$	$f_3:=x-z$
$\beta_2=0$	$f_4:=w$
$\beta_3=0$	$f_5:=z-1$
$\beta_4=0$	$f_6:=z$

We note that when $\alpha_2=0$, we see that the system (1) admits a particular solution $y=0$, and when $\alpha_4=\beta_4$, after we make the birational and symplectic transformation:

$$x_3=x-z,\ y_3=y,\ z_3=z,\ w_3=w+y \tag{13}$$

we see that the system (1) admits a particular solution $x_3=0$.

Remark 2.5. It is easy to see that the generators π_2,π_3,π_4 satisfy the relation:

$$\pi_4=\pi_2\pi_3\pi_2.$$

Remark 2.6. Taking the coordinate system

$$(X,Y,Z,W)=(1/x,-x(xy+\alpha_2),z,w),$$

it is easy to see that the transformation s_1 can be explicitly written as follows:

$$\begin{aligned}s_1:&(X,Y,Z,W,t;\alpha_0,\alpha_1,\alpha_2,\gamma_1,\beta_2,\beta_3,\beta_4)\\ &\rightarrow(X,Y-\alpha_1/X,Z,W,t;\alpha_0,-\alpha_1,\alpha_1+\alpha_2,\gamma_1,\beta_2,\beta_3,\beta_4).\end{aligned}$$

Proposition 2.7. *Let us define the following translation operators*

$$T_1:=\pi_1 s_5s_4s_3s_2s_1s_0s_2s_3s_4s_5,\quad T_2:=s_4s_6T_1s_6s_4,\quad T_3:=s_6T_1s_6,$$
$$T_4:=\pi_2T_1\pi_2,\quad T_5:=\pi_2T_2\pi_2,\quad T_6:=\pi_2T_3\pi_2.$$

These translation operators act on parameters $\alpha_i,\gamma_1,\beta_j$ as follows:

$$\begin{aligned}
T_1(\alpha_0,\alpha_1,\alpha_2,\gamma_1,\beta_2,\beta_3,\beta_4)&=(\alpha_0,\alpha_1,\alpha_2,\gamma_1,\beta_2,\beta_3,\beta_4)+(0,0,0,0,0,-1,1),\\
T_2(\alpha_0,\alpha_1,\alpha_2,\gamma_1,\beta_2,\beta_3,\beta_4)&=(\alpha_0,\alpha_1,\alpha_2,\gamma_1,\beta_2,\beta_3,\beta_4)+(0,0,0,1,-1,0,0),\\
T_3(\alpha_0,\alpha_1,\alpha_2,\gamma_1,\beta_2,\beta_3,\beta_4)&=(\alpha_0,\alpha_1,\alpha_2,\gamma_1,\beta_2,\beta_3,\beta_4)+(0,0,0,0,1,-1,-1),\\
T_4(\alpha_0,\alpha_1,\alpha_2,\gamma_1,\beta_2,\beta_3,\beta_4)&=(\alpha_0,\alpha_1,\alpha_2,\gamma_1,\beta_2,\beta_3,\beta_4)+(-1,1,0,0,0,0,0),\\
T_5(\alpha_0,\alpha_1,\alpha_2,\gamma_1,\beta_2,\beta_3,\beta_4)&=(\alpha_0,\alpha_1,\alpha_2,\gamma_1,\beta_2,\beta_3,\beta_4)+(0,0,-1,1,0,0,0),\\
T_6(\alpha_0,\alpha_1,\alpha_2,\gamma_1,\beta_2,\beta_3,\beta_4)&=(\alpha_0,\alpha_1,\alpha_2,\gamma_1,\beta_2,\beta_3,\beta_4)+(-1,-1,1,0,0,0,0).
\end{aligned}$$

Here, let us explain Figure 1:

(1) The above pictures of Figure 1 denote the Dynkin diagram of type $D_4^{(1)}$. The symbol in each circle denotes the invariant divisor of the sixth Painlevé system.

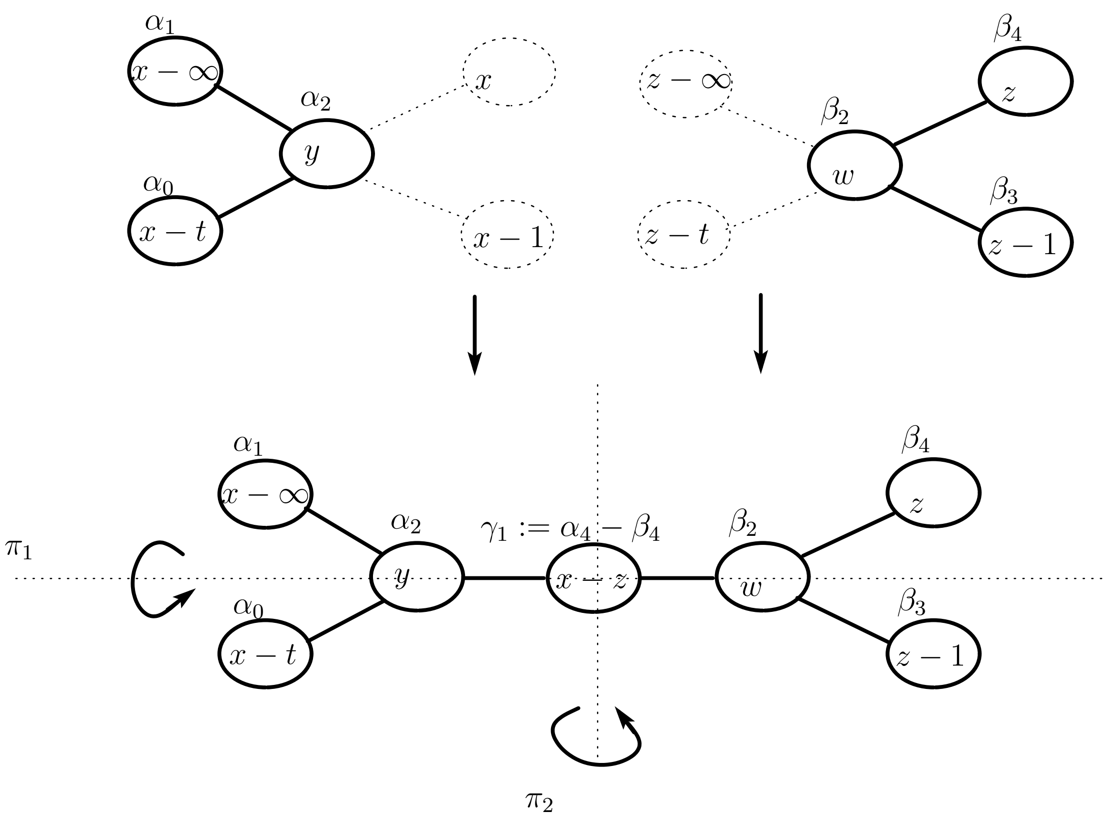

FIGURE 1. Dynkin diagram of type $D_6^{(1)}$

(2) The below picture of Figure 1 denotes the Dynkin diagram of type $D_6^{(1)}$. The symbol in each circle denotes the invariant divisor of the system (1).

Here, let us consider the following problem.

PROBLEM 2.8. Can we find an algebraic ordinary differential system with the following assumptions (A1),(A2)?

ASSUMPTION 2.9. $(A1)$ The system is a polynomial Hamiltonian system with the Hamiltonian $H \in \mathbb{C}(t)[x, y, z, w, \alpha_0, \alpha_1, \dots, \alpha_6]$.
$(A2)$ The system is invariant under the transformations $s_0, s_1, \dots, s_6$ given in Theorem 2.3.

In the brute force approach to Problem 2.8, we must deal with polynomials H in

$$\alpha_0, \alpha_1, \dots, \alpha_6, t, x, y, z, w.$$

This approach, however, soon came to a deadlock because of technical difficulties. In this paper, we present a new approach from the viewpoint of holomorphy.

THEOREM 2.10. *Let us consider a polynomial Hamiltonian system with Hamiltonian* $H \in \mathbb{C}[x, y, z, w]$. *We assume that*
$(A1)$ $deg(H) = 5$ *with respect to* x, y, z, w.

$(A2)$ *This system becomes again a polynomial Hamiltonian system in each coordinate system* (x_i, y_i, z_i, w_i) $(i = 0, 1, \dots, 6)$:

$$
\begin{aligned}
&r_0 : x_0 = -((x-t)y-\alpha_0)y,\ y_0 = 1/y,\ z_0 = z,\ w_0 = w,\\
&r_1 : x_1 = 1/x,\ y_1 = -x(xy+\alpha_1+\alpha_2),\ z_1 = z,\ w_1 = w,\\
&r_2 : x_2 = 1/x,\ y_2 = -x(xy+\alpha_2),\ z_2 = z,\ w_2 = w,\\
&r_3 : x_3 = -((x-z)y-\alpha_3)y,\ y_3 = 1/y,\ z_3 = z,\ w_3 = y+w,\\
&r_4 : x_4 = x,\ y_4 = y,\ z_4 = 1/z,\ w_4 = -z(zw+\alpha_4),\\
&r_5 : x_5 = x,\ y_5 = y,\ z_5 = -((z-1)w-\alpha_5)w,\ w_5 = 1/w,\\
&r_6 : x_6 = x,\ y_6 = y,\ z_6 = -w(zw-\alpha_6),\ w_6 = 1/w.
\end{aligned}
$$

Then such a system coincides with the system (1) *with the polynomial Hamiltonian*

$$
\begin{aligned}
H =\ &H_{VI}(x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_3 + \alpha_6)\\
&+ H_{VI}(z, w, t; \alpha_0 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_4, \alpha_5, \alpha_6) + \frac{2(x-t)yz\{(z-1)w+\alpha_4\}}{t(t-1)}.
\end{aligned}
$$

By this theorem, we can also recover the parameter's relation (3).

We note that the condition $(A2)$ should be read that

$$r_j(H) \quad (j = 1, 2, \dots, 6), \quad r_0(H - y)$$

are polynomials with respect to x_i, y_i, z_i, w_i.

In this method, $\alpha_0, \alpha_1, \dots, \alpha_6$ can be treated as parameters rather than variables. In the holomorphy requirement, we only need to consider polynomials in x, y, z, w. Hence, the number of unknown coefficients can be drastically reduced.

Theorems 2.3, 2.10 and Proposition 2.7 can be cheched by a direct calculation, respectively.

In addition to Theorems 2.3 and 2.10, we give an explicit description of a confluence process to the system of type $A_5^{(1)}$.

THEOREM 2.11. *For the system* (1) *of type* $D_6^{(1)}$, *we make the change of parameters and variables*

$$
\begin{aligned}
&\alpha_0 = \varepsilon^{-1}, \quad \alpha_1 = A_0, \quad \alpha_2 = A_1, \quad \alpha_4 - \beta_4 = A_2,\\
&\beta_2 = A_3, \quad \beta_3 = -\varepsilon^{-1} - (A_1 + A_2 + A_3 - A_5), \quad \beta_4 = A_4,
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&t = 1 - \varepsilon T, \quad x = \frac{X}{X-T}, \quad z = \frac{Z}{Z-T},\\
&y = -\frac{(X-T)\{(X-T)Y+A_1\}}{T}, \quad w = -\frac{(Z-T)\{(Z-T)W+A_3\}}{T}
\end{aligned}
\tag{15}
$$

from $\alpha_0, \alpha_1, \alpha_2, \gamma_1, \beta_2, \beta_3, \beta_4, t, x, y, z, w$ *to* $A_0, \dots, A_5, \varepsilon, T, X, Y, Z, W$. *Then the system* (1) *can also be written in the new variables* T, X, Y, Z, W *and parameters* $A_0, A_1, \dots, A_5, \varepsilon$ *as a Hamiltonian system. This new system tends to the system of type* $A_5^{(1)}$ *as* $\varepsilon \to 0$.

Here, the system of type $A_5^{(1)}$ is explicitly given in Section 6.

By proving the following theorem, we see how the transformation in Theorem 2.11 works on the Bäcklund transformation group $W(D_6^{(1)}) =< s_0, s_1, \dots, s_6 >$ described in Theorem 2.3.

THEOREM 2.12. *For the transformations* (14), (15) *given in Theorem 2.11 we can choose a subgroup* $W_{D_6^{(1)} \to A_5^{(1)}}$ *of the Bäcklund transformation group* $W(D_6^{(1)})$ *so that* $W_{D_6^{(1)} \to A_5^{(1)}}$ *converges to the Bäcklund transformation group* $W(A_5^{(1)})$ *of the system* (36) *with the polynomial Hamiltonian* (37) (see Section 6) .

3. THE SYSTEMS OF TYPE $B_6^{(1)}$

In this section, we find two types of a 6-parameter family of coupled Painlevé VI systems in dimension four with affine Weyl group symmetry of type $B_6^{(1)}$. Each of them is equivalent to a polynomial Hamiltonian system, however, each has a different represen- taion of type $B_6^{(1)}$. We also show that each of them is equivalent to the system (1) by a birational and symplectic transformation.

The first member is given by

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dz}{dt} = \frac{\partial H}{\partial w}, \quad \frac{dw}{dt} = -\frac{\partial H}{\partial z} \tag{16}$$

with the polynomial Hamiltonian

$$\begin{aligned} H = &\tilde{H}_{VI}(x, y, t; 2\alpha_0 + \alpha_1, \alpha_1, \alpha_2, \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_3 + \alpha_6) \\ &+ H_{VI}(z, w, t; 2\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_3, \alpha_4, \alpha_5, \alpha_6) \\ &\qquad + \frac{2xz\{(tx-1)y + t\alpha_2\}\{(z-1)w + \alpha_4\}}{t(t-1)}. \end{aligned} \tag{17}$$

Here x, y, z and w denote unknown complex variables, and $\alpha_0, \alpha_1, \dots, \alpha_6$ are complex parameters satisfying the relation

$$2\alpha_0 + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 = 1. \tag{18}$$

The symbol $\tilde{H}_{VI}(Q, P, t; \gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ denotes the Hamiltonian of the second-order Painlevé VI equation given by

$$\begin{aligned} &\tilde{H}_{VI}(Q, P, t; \gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \\ &= \frac{1}{t(t-1)}[P^2(tQ-1)(Q-1)Q - \{(\gamma_0 - 1)t(Q-1)Q + \gamma_1(Q-1)(tQ-1) \\ &+ \gamma_3 Q(tQ-1)\}P + \gamma_2(\gamma_2 + \gamma_4)tQ] \quad (\gamma_0 + \gamma_1 + 2\gamma_2 + \gamma_3 + \gamma_4 = 1). \end{aligned} \tag{19}$$

THEOREM 3.1. *The system* (16) *admits extended affine Weyl group symmetry of type* $B_6^{(1)}$ *as the group of its Bäcklund transformations, whose generators* $S_0, S_1, \dots, S_6, \varphi$ *are explicitly given as follows*: *with the notation*: $(*) := (x, y, z, w, t; \alpha_0, \alpha_1, \dots, \alpha_6)$,

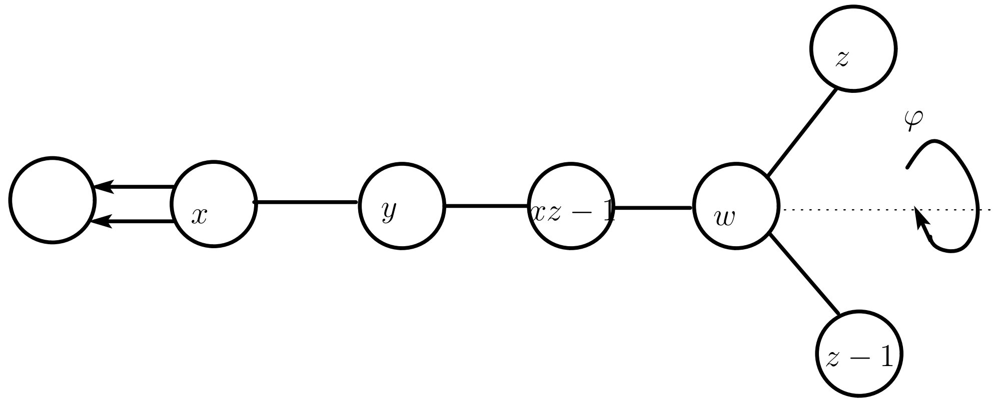

FIGURE 2. Dynkin diagram of type $B_6^{(1)}$

$$
\begin{aligned}
S_0 &: (*) \rightarrow (\frac{tx-1}{t-1}, \frac{(t-1)y}{t}, \frac{(t-1)z}{t-z}, \frac{(t-z)(tw-zw-\alpha_4)}{t(t-1)}, 1-t;\\
&\quad -\alpha_0, \alpha_1+2\alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6),\\
S_1 &: (*) \rightarrow \left(x, y-\frac{\alpha_1}{x}, z, w, t; \alpha_0+\alpha_1, -\alpha_1, \alpha_2+\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\right),\\
S_2 &: (*) \rightarrow \left(x+\frac{\alpha_2}{y}, y, z, w, t; \alpha_0, \alpha_1+\alpha_2, -\alpha_2, \alpha_3+\alpha_2, \alpha_4, \alpha_5, \alpha_6\right),\\
S_3 &: (*) \rightarrow \left(x, y-\frac{\alpha_3 z}{xz-1}, z, w-\frac{\alpha_3 x}{xz-1}, t; \alpha_0, \alpha_1, \alpha_2+\alpha_3, -\alpha_3, \alpha_4+\alpha_3, \alpha_5, \alpha_6\right),\\
S_4 &: (*) \rightarrow \left(x, y, z+\frac{\alpha_4}{w}, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3+\alpha_4, -\alpha_4, \alpha_5+\alpha_4, \alpha_6+\alpha_4\right),\\
S_5 &: (*) \rightarrow \left(x, y, z, w-\frac{\alpha_5}{z-1}, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4+\alpha_5, -\alpha_5, \alpha_6\right),\\
S_6 &: (*) \rightarrow \left(x, y, z, w-\frac{\alpha_6}{z}, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4+\alpha_6, \alpha_5, -\alpha_6\right),\\
\varphi &: (*) \rightarrow (\frac{x}{x-1}, -(x-1)\{(x-1)y+\alpha_2\}, 1-z, -w, 1-t;\\
&\quad \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_5).
\end{aligned}
$$

We remark that the Bäcklund transformations $S_1, \dots, S_6$ satisfy the relation (8). However, the transformation S_0 do not satisfy so.

Theorem 3.1 can be cheched by a direct calculation.

THEOREM 3.2. *For the system* (1) *of type* $D_6^{(1)}$, *we make the change of parameters and variables*

$$
\begin{aligned}
&\frac{(\alpha_0-\alpha_1)}{2} = A_0, \quad \alpha_1 = A_1, \quad \alpha_2 = A_2, \quad \alpha_4-\beta_4 = A_3,\\
&\beta_2 = A_4, \quad \beta_3 = A_5, \quad \beta_4 = A_6,
\end{aligned} \tag{20}
$$

$$
X = \frac{1}{x}, \quad Y = -(xy+\alpha_2)x, \quad Z = z, \quad W = w \tag{21}
$$

from $\alpha_0, \alpha_1, \alpha_2, \alpha_4, \beta_2, \beta_3, \beta_4, x, y, z, w$ *to* $A_0, A_1, \dots, A_6, X, Y, Z, W$. *Then the system* (1) *can also be written in the new variables* X, Y, Z, W *and parameters* $A_0, \dots, A_6$ *as a Hamiltonian system. This new system tends to the system* (16) *with the Hamiltonian* (17).

PROOF. Notice that

$$2A_0 + 2A_1 + 2A_2 + 2A_3 + 2A_4 + A_5 + A_6 = \alpha_0 + \alpha_1 + 2\alpha_2 + 2(\alpha_4 - \beta_4) + 2\beta_2 + \beta_3 + \beta_4 = 1$$

and the change of variables from (x, y, z, w) to (X, Y, Z, W) is symplectic. Choose S_i $(i = 0, 1, \dots, 6)$ and φ as

$$S_0 := \pi_4,\ S_1 := s_1,\ S_2 := s_2,\ S_3 := s_3,\ S_4 := s_4,\ S_5 := s_5,\ S_6 := s_6,\ \varphi := \pi_3.$$

Then the transformations S_i are reflections of the parameters $A_0, A_1, \dots, A_6$. The transformation group $\tilde{W}(B_6^{(1)}) =< S_0, S_1, \dots, S_6, \varphi >$ coincides with the transformations given in Theorem 3.1. $\square$

The second member is given by

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dz}{dt} = \frac{\partial H}{\partial w}, \quad \frac{dw}{dt} = -\frac{\partial H}{\partial z} \tag{22}$$

with the polynomial Hamiltonian

$$\begin{aligned} H = &H_{VI}(x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + \alpha_5 + 2\alpha_6, \alpha_3 + 2\alpha_4 + \alpha_5) \\ &+ \tilde{H}_{VI}(z, w, t; \alpha_0 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_4, \alpha_5 + 2\alpha_6, \alpha_5) \\ &+ \frac{2(x-t)y(z-1)w}{t(t-1)}. \end{aligned} \tag{23}$$

Here x, y, z and w denote unknown complex variables, and $\alpha_0, \alpha_1, \dots, \alpha_6$ are complex parameters satisfying the relation

$$\alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 = 1. \tag{24}$$

THEOREM 3.3. *The system* (22) *admits extended affine Weyl group symmetry of type* $B_6^{(1)}$ *as the group of its Bäcklund transformations, whose generators* $w_0, w_1, w_2, \dots, w_6, \psi$ *are explicitly given as follows*: *with the notation*: $(*) := (x, y, z, w, t; \alpha_0, \alpha_1, \dots, \alpha_6)$,

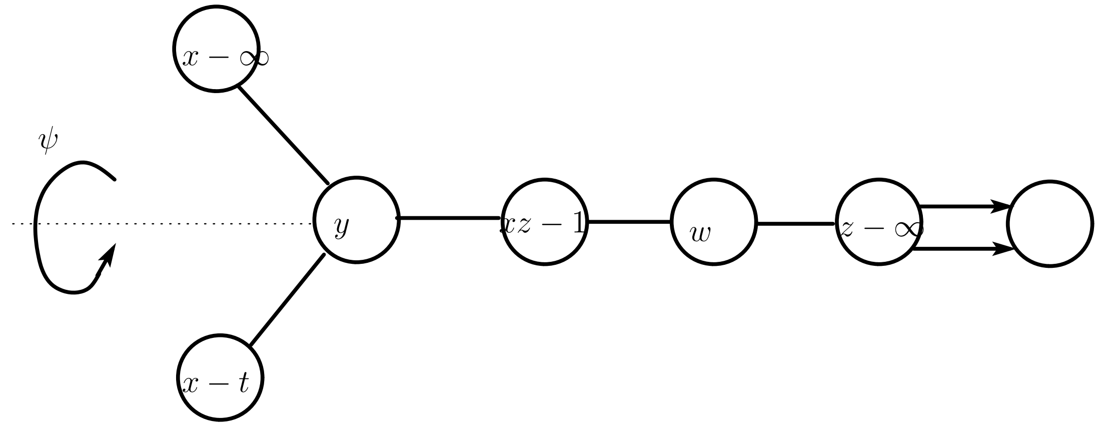

FIGURE 3. Dynkin diagram of type $B_6^{(1)}$

$$
\begin{aligned}
&w_0 : (*) \rightarrow \left(x, y - \frac{\alpha_0}{x-t}, z, w, t; -\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3, \alpha_4, \alpha_5, \alpha_6\right),\\
&w_1 : (*) \rightarrow (x, y, z, w, t; \alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6),\\
&w_2 : (*) \rightarrow \left(x + \frac{\alpha_2}{y}, y, z, w, t; \alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4, \alpha_5, \alpha_6\right),\\
&w_3 : (*) \rightarrow \left(x, y - \frac{\alpha_3 z}{xz-1}, z, w - \frac{\alpha_3 x}{xz-1}, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3, \alpha_5, \alpha_6\right),\\
&w_4 : (*) \rightarrow \left(x, y, z + \frac{\alpha_4}{w}, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + \alpha_4, -\alpha_4, \alpha_5 + \alpha_4, \alpha_6\right),\\
&w_5 : (*) \rightarrow (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 + \alpha_5, -\alpha_5, \alpha_6 + \alpha_5),\\
&w_6 : (*) \rightarrow (1-x, -y, \frac{z}{z-1}, -(z-1)\{(z-1)w + \alpha_4\}, 1-t;\\
&\qquad \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 + 2\alpha_6, -\alpha_6),\\
&\psi : (*) \rightarrow (\frac{(t-1)x}{t-x}, \frac{(t-x)\{(t-x)y - \alpha_2\}}{t(t-1)}, \frac{tz-1}{t-1}, \frac{(t-1)w}{t}, 1-t;\\
&\qquad \alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6).
\end{aligned}
$$

We remark that the Bäcklund transformations $w_0, \dots, w_5$ satisfy the relation (8). However, the transformation w_6 do not satisfy so.

Theorem 3.3 can be cheched by a direct calculation.

THEOREM 3.4. *For the system* (1) *of type* $D_6^{(1)}$, *we make the change of parameters and variables*

$$
\alpha_0 = A_0, \quad \alpha_1 = A_1, \quad \alpha_2 = A_2, \quad \alpha_4 - \beta_4 = A_3,
$$
$$
\beta_2 = A_4, \quad \beta_4 = A_5, \quad \frac{(\beta_3 - \beta_4)}{2} = A_6, \tag{25}
$$

$$
X = x, \quad Y = y, \quad Z = \frac{1}{z}, \quad W = -z(zw + \beta_2) \tag{26}
$$

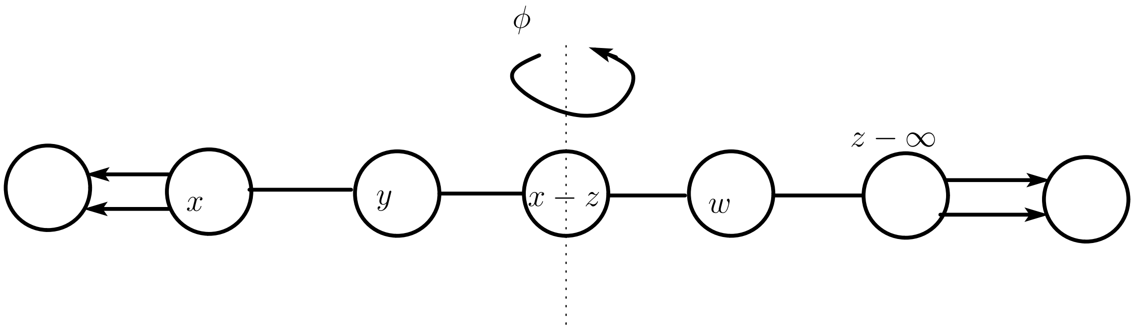

FIGURE 4. Dynkin diagram of type $D_7^{(2)}$

from $\alpha_0, \alpha_1, \alpha_2, \alpha_4, \beta_2, \beta_3, \beta_4, x, y, z, w$ *to* $A_0, A_1, \dots, A_6, X, Y, Z, W$. *Then the system* (1) *can also be written in the new variables* X, Y, Z, W *and parameters* $A_0, A_1, \dots, A_6$ *as a Hamiltonian system. This new system tends to the system* (22) *with the Hamiltonian* (23).

Proof. Notice that

$$A_0 + A_1 + 2A_2 + 2A_3 + 2A_4 + 2A_5 + 2A_6 = \alpha_0 + \alpha_1 + 2\alpha_2 + 2(\alpha_4 - \beta_4) + 2\beta_2 + \beta_3 + \beta_4 = 1$$

and the change of variables from (x, y, z, w) to (X, Y, Z, W) is symplectic. Choose w_i $(i = 0, 1, \dots, 6)$ and ψ as

$$w_0 := s_0, \ w_1 := s_1, \ w_2 := s_2, \ w_3 := s_3, \ w_4 := s_4, \ w_5 := s_6, \ w_6 := \pi_3, \ \psi := \pi_4.$$

Then the transformations w_i are reflections of the parameters $A_0, A_1, \dots, A_6$. The transformation group $\tilde{W}(B_6^{(1)}) =< w_0, w_1, \dots, w_6, \psi >$ coincides with the transformations given in Theorem 3.3. $\square$

4. THE SYSTEM OF TYPE $D_7^{(2)}$

In this section, we find a 6-parameter family of coupled Painlevé VI systems in dimension four with affine Weyl group symmetry of type $D_7^{(2)}$. This system is equivalent to a polynomial Hamiltonian system. In the final stage of this section, this system is equivalent to the system (1) by a birational and symplectic transformation.

This system is explicitly given as follows:

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dz}{dt} = \frac{\partial H}{\partial w}, \quad \frac{dw}{dt} = -\frac{\partial H}{\partial z} \tag{27}$$

with the polynomial Hamiltonian

$$\begin{aligned} H = &\tilde{H}_{VI}(x, y, t; 2\alpha_0 + \alpha_1, \alpha_1, \alpha_2, \alpha_3 + \alpha_5 + 2\alpha_6, \alpha_3 + 2\alpha_4 + \alpha_5) \\ &+ \tilde{H}_{VI}(z, w, t; 2\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_3, \alpha_4, \alpha_5 + 2\alpha_6, \alpha_5) \\ &+ \frac{2x\{(tx - 1)y + t\alpha_2\}(z - 1)w}{t(t - 1)}. \end{aligned} \tag{28}$$

Here x, y, z and w denote unknown complex variables, and $\alpha_0, \alpha_1, \dots, \alpha_6$ are complex parameters satisfying the relation

$$\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6=\frac{1}{2}. \tag{29}$$

THEOREM 4.1. *The system* (27) *admits extended affine Weyl group symmetry of type* $D_7^{(2)}$ *as the group of its Bäcklund transformations, whose generators* $u_0, u_1, \dots, u_6, \phi$ *are explicitly given as follows*: *with the notation*: $(*) := (x, y, z, w, t; \alpha_0, \alpha_1, \dots, \alpha_6)$,

$$\begin{aligned}
u_0 : (*) \rightarrow &(\frac{tx-1}{t-1}, \frac{(t-1)y}{t}, \frac{tz-1}{t-1}, \frac{(t-1)w}{t}, 1-t;\\
&-\alpha_0, \alpha_1+2\alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6),\\
u_1 : (*) \rightarrow &\left(x, y-\frac{\alpha_1}{x}, z, w, t; \alpha_0+\alpha_1, -\alpha_1, \alpha_2+\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\right),\\
u_2 : (*) \rightarrow &\left(x+\frac{\alpha_2}{y}, y, z, w, t; \alpha_0, \alpha_1+\alpha_2, -\alpha_2, \alpha_3+\alpha_2, \alpha_4, \alpha_5, \alpha_6\right),\\
u_3 : (*) \rightarrow &\left(x, y-\frac{\alpha_3}{x-z}, z, w+\frac{\alpha_3}{x-z}, t; \alpha_0, \alpha_1, \alpha_2+\alpha_3, -\alpha_3, \alpha_4+\alpha_3, \alpha_5, \alpha_6\right),\\
u_4 : (*) \rightarrow &\left(x, y, z+\frac{\alpha_4}{w}, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3+\alpha_4, -\alpha_4, \alpha_5+\alpha_4, \alpha_6\right),\\
u_5 : (*) \rightarrow &(x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4+\alpha_5, -\alpha_5, \alpha_6+\alpha_5),\\
u_6 : (*) \rightarrow &(\frac{x}{x-1}, -(x-1)\{(x-1)y+\alpha_2\}, \frac{z}{z-1}, -(z-1)\{(z-1)w+\alpha_4\}, 1-t;\\
&\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5+2\alpha_6, -\alpha_6),\\
\phi : (*) \rightarrow &\left(\frac{1}{tz}, -tz(zw+\alpha_4), \frac{1}{tx}, -tx(xy+\alpha_2), t; \alpha_6, \alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0\right).
\end{aligned}$$

We remark that the Bäcklund transformations $u_1, \dots, u_5$ satisfy the relation (8). However, the transformations u_0, u_6 do not satisfy so.

Theorem 4.1 can be cheched by a direct calculation.

THEOREM 4.2. *For the system* (1) *of type* $D_6^{(1)}$, *we make the change of parameters and variables*

$$\begin{aligned}
&\frac{(\alpha_0-\alpha_1)}{2}=A_0, \quad \alpha_1=A_1, \quad \alpha_2=A_2, \quad \alpha_4-\beta_4=A_3,\\
&\beta_2=A_4, \quad \beta_4=A_5, \quad \frac{(\beta_3-\beta_4)}{2}=A_6,
\end{aligned} \tag{30}$$

$$X=\frac{1}{x}, \quad Y=-(xy+\alpha_2)x, \quad Z=\frac{1}{z}, \quad W=-(zw+\beta_2)z \tag{31}$$

from $\alpha_0, \alpha_1, \alpha_2, \alpha_4, \beta_2, \beta_3, \beta_4, x, y, z, w$ *to* $A_0, A_1, \dots, A_6, X, Y, Z, W$. *Then the system* (1) *can also be written in the new variables* X, Y, Z, W *and parameters* $A_0, \dots, A_6$ *as a Hamiltonian system. This new system tends to the system* (27) *with the Hamiltonian* (28).

Proof. Notice that

$$2(A_0+A_1+A_2+A_3+A_4+A_5+A_6)=\alpha_0+\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6=1$$

and the change of variables from (x,y,z,w) to (X,Y,Z,W) is symplectic. Choose u_i $(i=0,1,\dots,6)$ and ϕ as

$$u_0:=\pi_4,\ u_1:=s_1,\ u_2:=s_2,\ u_3:=s_3,\ u_4:=s_4,\ u_5:=s_6,\ u_6:=\pi_3,\ \phi:=\pi_2.$$

Then the transformations u_i are reflections of the parameters $A_0,A_1,\dots,A_6$. The transformation group $\tilde{W}(D_7^{(2)})=<u_0,u_1,\dots,u_6,\phi>$ coincides with the transformations given in Theorem 4.1. □

Finally, let us summarize Sections 3 and 4 in the below figure.

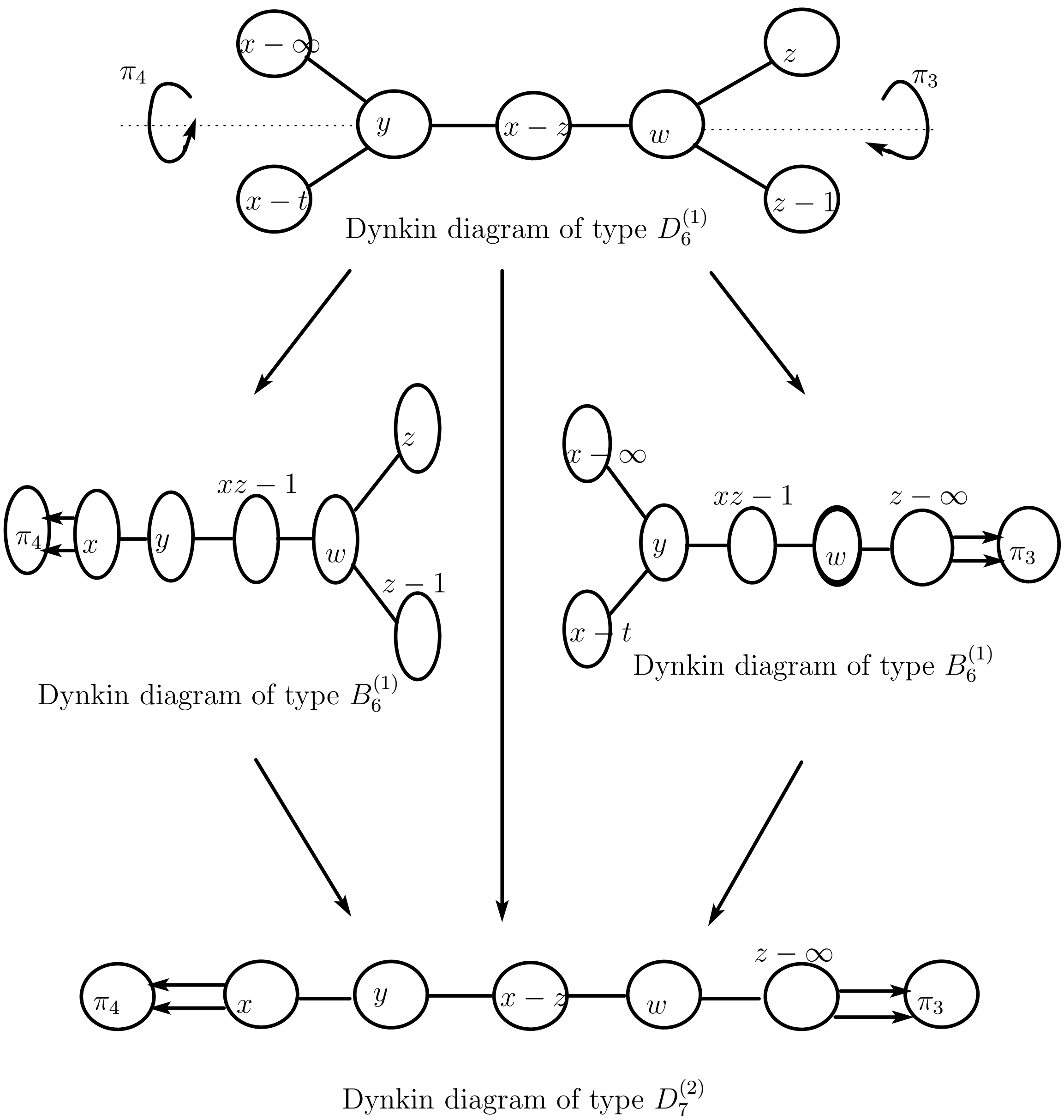

FIGURE 5. Dynkin diagrams of types $B_6^{(1)}$,$D_6^{(1)}$ and $D_7^{(2)}$

5. Autonomous version of type $D_6^{(1)}$

In this section, we find an autonomous version of the system (1) explicitly given by

$$\frac{dx}{dt}=\frac{\partial H}{\partial y}, \quad \frac{dy}{dt}=-\frac{\partial H}{\partial x}, \quad \frac{dz}{dt}=\frac{\partial H}{\partial w}, \quad \frac{dw}{dt}=-\frac{\partial H}{\partial z} \tag{32}$$

with the polynomial Hamiltonian

$$\begin{aligned} H=&H_{auto}(x,y;\alpha_0,\alpha_1,\alpha_2,\alpha_3+2\alpha_4+\alpha_5,\alpha_3+\alpha_6)\\ &+H_{auto}(z,w;\alpha_0+\alpha_3,\alpha_1+2\alpha_2+\alpha_3,\alpha_4,\alpha_5,\alpha_6)\\ &+2(x-\eta)yz\{(z-1)w+\alpha_4\}. \end{aligned} \tag{33}$$

Here x,y,z and w denote unknown complex variables, and η and $\alpha_0,\alpha_1,\dots,\alpha_6$ are complex parameters satisfying the relation:

$$\alpha_0+\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6=0. \tag{34}$$

Here the symbol $H_{auto}(q,p;\beta_0,\beta_1,\beta_2,\beta_3,\beta_4)$ denotes the Hamiltonian given by

$$\begin{aligned} &H_{auto}(q,p;\beta_0,\beta_1,\beta_2,\beta_3,\beta_4)=\\ &q(q-1)(q-\eta)p^2-\{\beta_0q(q-1)+\beta_4(q-1)(q-\eta)+\beta_3q(q-\eta)\}p+\beta_2(\beta_1+\beta_2)q\\ &\qquad\qquad(\beta_0+\beta_1+2\beta_2+\beta_3+\beta_4=0). \end{aligned} \tag{35}$$

Proposition 5.1. *The system* (32) *has the Hamiltonian* (33) *as its first integral.*

Theorem 5.2. *The system* (32) *admits the affine Weyl group symmetry of type $D_6^{(1)}$ as the group of its Bäcklund transformations, whose generators $s_0,s_1,\dots,s_6$ are explicitly given as follows: with the notation:* $(*):=(x,y,z,w;\alpha_0,\alpha_1,\dots,\alpha_6)$,

$$\begin{aligned} s_0:(*)\rightarrow&\left(x,y-\frac{\alpha_0}{x-\eta},z,w;-\alpha_0,\alpha_1,\alpha_2+\alpha_0,\alpha_3,\alpha_4,\alpha_5,\alpha_6\right),\\ s_1:(*)\rightarrow&(x,y,z,w;\alpha_0,-\alpha_1,\alpha_2+\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6),\\ s_2:(*)\rightarrow&\left(x+\frac{\alpha_2}{y},y,z,w;\alpha_0+\alpha_2,\alpha_1+\alpha_2,-\alpha_2,\alpha_3+\alpha_2,\alpha_4,\alpha_5,\alpha_6\right),\\ s_3:(*)\rightarrow&\left(x,y-\frac{\alpha_3}{x-z},z,w+\frac{\alpha_3}{x-z};\alpha_0,\alpha_1,\alpha_2+\alpha_3,-\alpha_3,\alpha_4+\alpha_3,\alpha_5,\alpha_6\right),\\ s_4:(*)\rightarrow&\left(x,y,z+\frac{\alpha_4}{w},w;\alpha_0,\alpha_1,\alpha_2,\alpha_3+\alpha_4,-\alpha_4,\alpha_5+\alpha_4,\alpha_6+\alpha_4\right),\\ s_5:(*)\rightarrow&\left(x,y,z,w-\frac{\alpha_5}{z-1};\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4+\alpha_5,-\alpha_5,\alpha_6\right),\\ s_6:(*)\rightarrow&\left(x,y,z,w-\frac{\alpha_6}{z};\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4+\alpha_6,\alpha_5,-\alpha_6\right). \end{aligned}$$

We note that these transformations s_i are birational and symplectic.

THEOREM 5.3. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H \in \mathbb{C}[x,y,z,w]$. We assume that*

(A1) $deg(H) = 5$ with respect to x,y,z,w.

(A2) This system becomes again a polynomial Hamiltonian system in each coordinate system (x_i,y_i,z_i,w_i) $(i=0,1,\dots,6)$:

$$\begin{aligned}
&x_0 = -((x-\eta)y-\alpha_0)y, \quad y_0 = 1/y, \quad z_0 = z, \quad w_0 = w,\\
&x_1 = 1/x, \quad y_1 = -x(xy+\alpha_1+\alpha_2), \quad z_1 = z, \quad w_1 = w,\\
&x_2 = 1/x, \quad y_2 = -x(xy+\alpha_2), \quad z_2 = z, \quad w_2 = w,\\
&x_3 = -((x-z)y-\alpha_3)y, \quad y_3 = 1/y, \quad z_3 = z, \quad w_3 = y+w,\\
&x_4 = x, \quad y_4 = y, \quad z_4 = 1/z, \quad w_4 = -z(zw+\alpha_4),\\
&x_5 = x, \quad y_5 = y, \quad z_5 = -((z-1)w-\alpha_5)w, \quad w_5 = 1/w,\\
&x_6 = x, \quad y_6 = y, \quad z_6 = -w(zw-\alpha_6), \quad w_6 = 1/w.
\end{aligned}$$

Then such a system coincides with the system (32).

6. Review of the systems of types $A_4^{(1)}$ and $A_5^{(1)}$

Let us recall the system of type $A_5^{(1)}$ given by

(36)
$$\begin{cases}
\dfrac{dx}{dt} = \dfrac{\partial H_{A_5^{(1)}}}{\partial y} = \dfrac{2x^2y+2xzw}{t} - \dfrac{x^2}{t} - 2xy - 2zw + \left(1+\dfrac{\alpha_1+\alpha_3+\alpha_5}{t}\right)x + \alpha_2 + \alpha_4,\\
\dfrac{dy}{dt} = -\dfrac{\partial H_{A_5^{(1)}}}{\partial x} = \dfrac{-2xy^2-2yzw}{t} + y^2 + \dfrac{2xy}{t} - \left(1+\dfrac{\alpha_1+\alpha_3+\alpha_5}{t}\right)y + \dfrac{\alpha_1}{t},\\
\dfrac{dz}{dt} = \dfrac{\partial H_{A_5^{(1)}}}{\partial w} = \dfrac{2z^2w+2xyz}{t} - \dfrac{z^2}{t} - 2zw - 2yz + \left(1+\dfrac{\alpha_1+\alpha_3+\alpha_5}{t}\right)z + \alpha_4,\\
\dfrac{dw}{dt} = -\dfrac{\partial H_{A_5^{(1)}}}{\partial z} = \dfrac{-2zw^2-2xyw}{t} + w^2 + \dfrac{2zw}{t} + 2yw - \left(1+\dfrac{\alpha_1+\alpha_3+\alpha_5}{t}\right)w + \dfrac{\alpha_3}{t}
\end{cases}$$

with the polynomial Hamiltonian:

$$\begin{aligned}
H_{A_5^{(1)}} =& H_V(x,y,t;\alpha_1+\alpha_3+\alpha_5,\alpha_2+\alpha_4,\alpha_1)\\
&+ H_V(z,w,t;\alpha_1+\alpha_3+\alpha_5,\alpha_4,\alpha_3) - 2yzw + \frac{2xyzw}{t}.
\end{aligned} \tag{37}$$

Here, x,y,z and w denote unknown complex variables, and $\alpha_0,\alpha_1,\dots,\alpha_5$ are complex parameters with $\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 = 1$, and the symbol $H_V(q,p,t;\gamma_1,\gamma_2,\gamma_3)$ denotes the Hamiltonian of the second-order Painlevé V equation given by

$$H_V(q,p,t;\gamma_1,\gamma_2,\gamma_3) = \frac{q^2p^2-q^2p}{t} - qp^2 + \left(1+\frac{\gamma_3}{t}\right)qp + \gamma_2 p - \frac{\gamma_1 q}{t}. \tag{38}$$

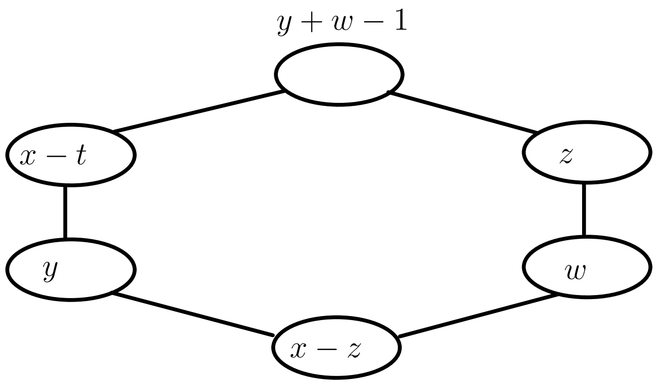

FIGURE 6. Dynkin diagram of type $A_5^{(1)}$

It is known that the system (36) admits the affine Weyl group symmetry of type $A_5^{(1)}$ as the group of its Bäcklund transformations, whose generators $s_0, s_1, \dots, s_5$ are explicitly given as follows: with the notation: $(*) := (x, y, z, w, t; \alpha_0, \alpha_1, \dots, \alpha_5)$,

$$\begin{aligned}
s_0 &: (*) \to \left(x, y - \frac{\alpha_0}{x-t}, z, w, t; -\alpha_0, \alpha_1 + \alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5 + \alpha_0\right),\\
s_1 &: (*) \to \left(x + \frac{\alpha_1}{y}, y, z, w, t; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4, \alpha_5\right),\\
s_2 &: (*) \to \left(x, y - \frac{\alpha_2}{x-z}, z, w + \frac{\alpha_2}{x-z}, t; \alpha_0, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4, \alpha_5\right),\\
s_3 &: (*) \to \left(x, y, z + \frac{\alpha_3}{w}, w, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3, \alpha_5\right),\\
s_4 &: (*) \to \left(x, y, z, w - \frac{\alpha_4}{z}, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + \alpha_4, -\alpha_4, \alpha_5 + \alpha_4\right),\\
s_5 &: (*) \to \left(x + \frac{\alpha_5}{y+w-1}, y, z + \frac{\alpha_5}{y+w-1}, w, t; \alpha_0 + \alpha_5, \alpha_1, \alpha_2, \alpha_3, \alpha_4 + \alpha_5, -\alpha_5\right).
\end{aligned}$$

THEOREM 6.1. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H \in C(t)[x, y, z, w]$. We assume that*

(A1) $deg(H) = 4$ with respect to x, y, z, w.

(A2) This system becomes again a polynomial Hamiltonian system in each coordinate system $r_i (i = 0, 1, \dots, 5)$:

$$\begin{aligned}
&r_0 : x_0 = -((x-t)y - \alpha_0)y, \quad y_0 = 1/y, \quad z_0 = z, \quad w_0 = w,\\
&r_1 : x_1 = 1/x, \quad y_1 = -(xy + \alpha_1)x, \quad z_1 = z, \quad w_1 = w,\\
&r_2 : x_2 = -((x-z)y - \alpha_2)y, \quad y_2 = 1/y, \quad z_2 = z, \quad w_2 = w + y,\\
&r_3 : x_3 = x, \quad y_3 = y, \quad z_3 = 1/z, \quad w_3 = -(zw + \alpha_3)z,\\
&r_4 : x_4 = x, \quad y_4 = y, \quad z_4 = -(zw - \alpha_4)w, \quad w_4 = 1/w,\\
&r_5 : x_5 = 1/x, \quad y_5 = -((y+w-1)y + \alpha_5)x, \quad z_5 = z - x, \quad w_5 = w.
\end{aligned}$$

Then such a system coincides with the system (36).

Next, let us recall the system of type $A_4^{(1)}$ given by

$$
(39)\qquad \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H_{A_4^{(1)}}}{\partial y} = x^2 + 2xy + 2zw - tx + \alpha_2 + \alpha_4, \\ \dfrac{dy}{dt} = -\dfrac{\partial H_{A_4^{(1)}}}{\partial x} = -y^2 - 2xy + ty + \alpha_1, \\ \dfrac{dz}{dt} = \dfrac{\partial H_{A_4^{(1)}}}{\partial w} = z^2 + 2zw + 2yz - tz + \alpha_4, \\ \dfrac{dw}{dt} = -\dfrac{\partial H_{A_4^{(1)}}}{\partial z} = -w^2 - 2zw - 2yw + tw + \alpha_3 \end{cases}
$$

with the polynomial Hamiltonian:

$$
(40)\qquad H_{A_4^{(1)}} = H_{IV}(x, y, t; \alpha_1, \alpha_2 + \alpha_4) + H_{IV}(z, w, t; \alpha_3, \alpha_4) + 2yzw.
$$

Here, x, y, z and w denote unknown complex variables, and $\alpha_0, \alpha_1, \dots, \alpha_4$ are complex parameters with $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$, and the symbol $H_{IV}(q, p, t; \gamma_1, \gamma_2)$ denotes the Hamiltonian of the second-order Painlevé IV equation given by

$$
(41)\qquad H_{IV}(q, p, t; \gamma_1, \gamma_2) = q^2p + qp^2 - tqp - \gamma_1 q + \gamma_2 p.
$$

It is known that the system (39) admits the affine Weyl group symmetry of type $A_4^{(1)}$ as the group of its Bäcklund transformations, whose generators $s_0, s_1, \dots, s_4$ are explicitly given as follows: with the notation: $(*) := (x, y, z, w, t; \alpha_0, \alpha_1, \dots, \alpha_4)$,

$$
\begin{aligned}
s_0 : (*) \rightarrow &(x + \frac{\alpha_0}{x + y + w - t}, y - \frac{\alpha_0}{x + y + w - t}, z + \frac{\alpha_0}{x + y + w - t}, w, t; \\
&- \alpha_0, \alpha_1 + \alpha_0, \alpha_2, \alpha_3, \alpha_4 + \alpha_0), \\
s_1 : (*) \rightarrow &\left(x + \frac{\alpha_1}{y}, y, z, w, t; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4\right), \\
s_2 : (*) \rightarrow &\left(x, y - \frac{\alpha_2}{x - z}, z, w + \frac{\alpha_2}{x - z}, t; \alpha_0, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4\right), \\
s_3 : (*) \rightarrow &\left(x, y, z + \frac{\alpha_3}{w}, w, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3\right), \\
s_4 : (*) \rightarrow &\left(x, y, z, w - \frac{\alpha_4}{z}, t; \alpha_0 + \alpha_4, \alpha_1, \alpha_2, \alpha_3 + \alpha_4, -\alpha_4\right).
\end{aligned}
$$

THEOREM 6.2. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H \in C(t)[x, y, z, w]$. We assume that*

(A1) $deg(H) = 3$ with respect to x, y, z, w.

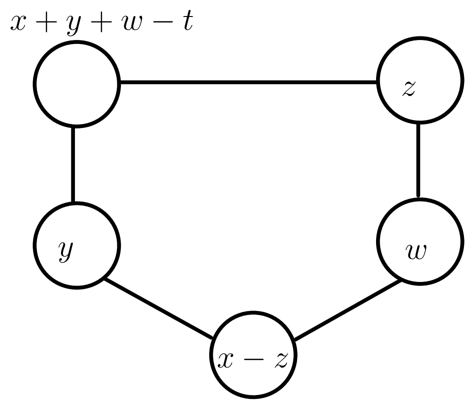

FIGURE 7. Dynkin diagram of type $A_4^{(1)}$

(*A*2) *This system becomes again a polynomial Hamiltonian system in each coordinate system* $r_i (i=0,1,\dots,4)$:

$$\begin{aligned}
&r_0 : x_0 = -((x+y+w-t)y-\alpha_0)y, \quad y_0 = 1/y, \quad z_0 = z+y, \quad w_0 = w,\\
&r_1 : x_1 = 1/x, \quad y_1 = -(xy+\alpha_1)x, \quad z_1 = z, \quad w_1 = w,\\
&r_2 : x_2 = -((x-z)y-\alpha_2)y, \quad y_2 = 1/y, \quad z_2 = z, \quad w_2 = w+y,\\
&r_3 : x_3 = x, \quad y_3 = y, \quad z_3 = 1/z, \quad w_3 = -(zw+\alpha_3)z,\\
&r_4 : x_4 = x, \quad y_4 = y, \quad z_4 = -(zw-\alpha_4)w, \quad w_4 = 1/w.
\end{aligned}$$

Then such a system coincides with the system (39).

Theorems 6.1 and 6.2 can be cheched by a direct calculation, respectively.

7. AN APPROACH FOR OBTAINING THE SYSTEM (1)

Much effort has been made to investigate algebraic ordinary differential systems with symmetry under the affine Weyl group of type $D_6^{(1)}$, however these systems have not yet been found. Taking a hint from the representation of the affine Weyl groups of types $A_4^{(1)}$ and $A_5^{(1)}$, we consider Problem 1. We do not yet have explicit descriptions of the symmetry under the affine Weyl group of type $D_6^{(1)}$ with respect to x, y, z, w, so we will construct a representation under the affine Weyl group of type $D_6^{(1)}$ by using a part of the symmetry under the affine Weyl groups of types $A_4^{(1)}$ and $A_5^{(1)}$. In the case of the Painlevé systems, the fourth, fifth and sixth Painlevé systems have affine Weyl group symmetry of type $A_2^{(1)}, A_3^{(1)}$ and $D_4^{(1)}$, respectively. Each of them has a common subgroup, which is isomorphic to the classical Weyl group $W(A_2)$. Here, the elements u_i of the subgroup

$W(A_2) =< u_1, u_2 >$ are explicitly written as follows:

$$u_1 : (x, y, \gamma_1, \gamma_2) \to \left(x + \frac{\gamma_1}{y}, y, -\gamma_1, \gamma_2 + \gamma_1\right),$$
$$u_2 : (x, y, \gamma_1, \gamma_2) \to \left(x, y - \frac{\gamma_2}{x}, \gamma_1 + \gamma_2, -\gamma_2\right).$$

Here, γ_1 and γ_2 are root parameters.

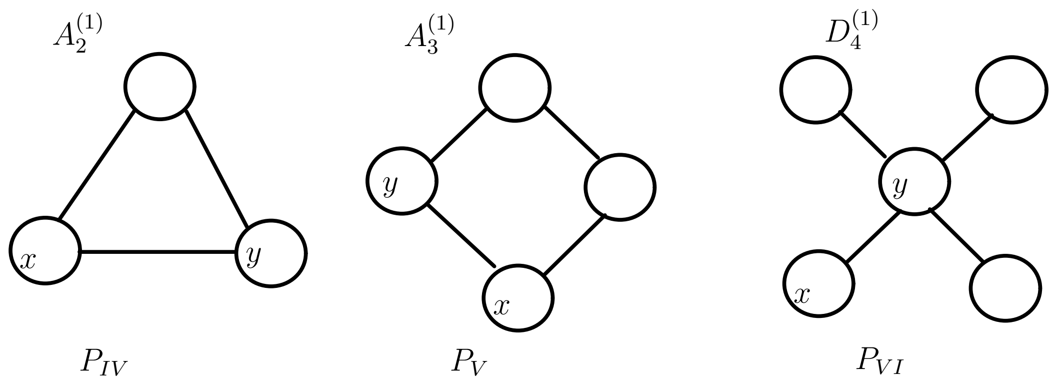

FIGURE 8. Dynkin diagrams of types $A_2^{(1)}, A_3^{(1)}$ and $D_4^{(1)}$

From the viewpoint of holomorphy, these transformations u_1, u_2 correspond to canonical coordinate systems (x_i, y_i) $(i = 1, 2)$ (see [13]), which are explicitly written as follows:

$$(x_1, y_1) = (1/x, -(xy + \gamma_1)x),$$
$$(x_2, y_2) = (-(xy - \gamma_2)y, 1/y).$$

These canonical coordinate systems can be obtained by successive blowing-up procedures at the beginning of the accessible singular points

$$P_1 = \{(X_1, Y_1) = (0, 0)\}, \quad P_2 = \{(X_2, Y_2) = (0, 0)\}$$

on the boundary divisor of $\mathbb{P}^2$. Here the coordinate systems (X_i, Y_i) are the boundary coordinate systems of $\mathbb{P}^2$ with the rational transformations

$$(X_1, Y_1) = (1/x, y/x), \qquad (X_2, Y_2) = (x/y, 1/y).$$

PROPOSITION 7.1. *Let us consider a polynomial Hamiltonian system with Hamiltonian* $K \in \mathbb{C}(t)[x, y]$. *We assume that*

(A1) $deg(K) = 5$ *with respect to* x, y.

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate systems* (x_i, y_i) $(i = 1, 2)$.

Then such a system is explicitly given as follows:

(42)
$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial K}{\partial y} = 2a_1x^3y + 3a_2x^2y^2 + 2a_3x^2y + a_4x^2 + 2a_5xy + a_6x - \gamma_2a_5 - \gamma_2^2a_2, \\ \dfrac{dy}{dt} = -\dfrac{\partial K}{\partial x} = -3a_1x^2y^2 - 2a_2xy^3 - 2a_3xy^2 - a_5y^2 - 2a_4xy - a_6y - \gamma_1a_4 + {\gamma_1}^2a_1 \end{cases}$$

with the polynomial Hamiltonian K

$$K = a_1x^3y^2 + a_2x^2y^3 + a_3x^2y^2 + a_4x^2y + a_5xy^2 + a_6xy - (\gamma_2 a_5 + \gamma_2^2 a_2)y + (\gamma_1 a_4 - {\gamma_1}^2 a_1)x. \tag{43}$$

Here, $a_1, a_2, \dots, a_6$ are undetermined rational functions in t.

In the case of dimension four, it is easy to see that the affine Weyl groups $W(A_5^{(1)})$ and $W(A_4^{(1)})$ have a common subgroup W, whose elements g_i are explicitly written as follows.

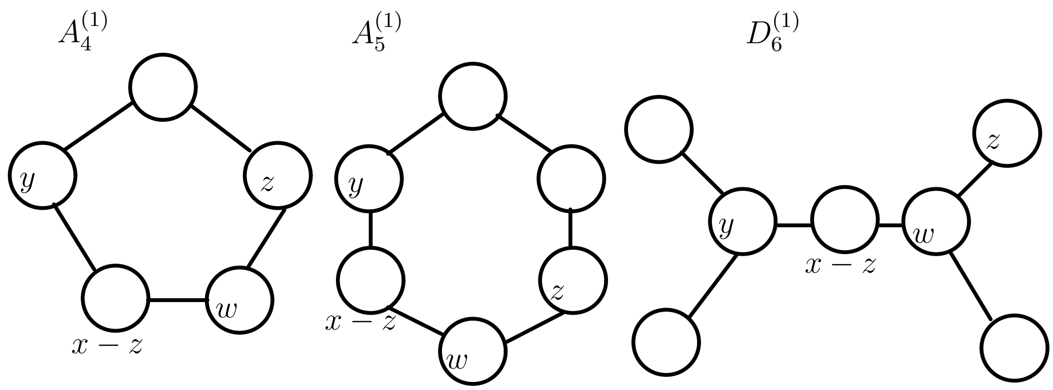

FIGURE 9. Dynkin diagrams of types $A_4^{(1)}, A_5^{(1)}$ and $D_6^{(1)}$

$$g_1 : (x, y, z, w, t, \alpha_1, \dots, \alpha_4) \to \left(x, y, z + \frac{\alpha_1}{w}, w, t, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4 + \alpha_1\right),$$
$$g_2 : (x, y, z, w, t, \alpha_1, \dots, \alpha_4) \to \left(x, y, z, w - \frac{\alpha_2}{z}, t, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3, \alpha_4\right),$$
$$g_3 : (x, y, z, w, t, \alpha_1, \dots, \alpha_4) \to \left(x + \frac{\alpha_3}{y}, y, z, w, t, \alpha_1, \alpha_2, -\alpha_3, \alpha_4 + \alpha_3\right),$$
$$g_4 : (x, y, z, w, t, \alpha_1, \dots, \alpha_4) \to \left(x, y - \frac{\alpha_4}{x - z}, z, w + \frac{\alpha_4}{x - z}, t; \alpha_1 + \alpha_4, \alpha_2, \alpha_3 + \alpha_4, -\alpha_4\right).$$

Here, $\alpha_1, \alpha_2, \alpha_3$ and α_4 are root parameters.

PROPOSITION 7.2. *The transformations g_i described above define a representation of the classical Weyl group of type A_4.*

PROPOSITION 7.3. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H \in \mathbb{C}(t)[x, y, z, w]$. We assume that*

(A1) $deg(H) = 5$ with respect to x, y, z, w.

(*A*2) *This system becomes again a polynomial Hamiltonian system in each coordinate systems* (x_i, y_i, z_i, w_i) $(i = 1, 2, 3, 4)$:

$$
\begin{aligned}
g_1 :& x_1 = x, \quad y_1 = y, \quad z_1 = 1/z, \quad w_1 = -z(zw + \alpha_1),\\
g_2 :& x_2 = x, \quad y_2 = y, \quad z_2 = -w(zw - \alpha_2), \quad w_2 = 1/w,\\
g_3 :& x_3 = 1/x, \quad y_3 = -x(xy + \alpha_3), \quad z_3 = z, \quad w_3 = w,\\
g_4 :& x_4 = -((x - z)y - \alpha_4)y, \quad y_4 = 1/y, \quad z_4 = z, \quad w_4 = y + w.
\end{aligned}
$$

Then such a system is explicitly given as follows:

$$
\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dz}{dt} = \frac{\partial H}{\partial w}, \quad \frac{dw}{dt} = -\frac{\partial H}{\partial z} \tag{44}
$$

with the polynomial Hamiltonian

(45)

$$
\begin{aligned}
H =& \frac{(b_5 + b_9)}{2}x^3y^2 + \frac{(b_4 - b_7)}{2}x^2y^3 + \frac{(b_1 + b_3)}{2}x^2y^2 + b_8x^2y + b_6xy^2 + b_2xy\\
&+ \frac{(3\alpha_1\alpha_2 + \alpha_2^2 + 2\alpha_2\alpha_4 + \alpha_4^2)b_4 + 2(-\alpha_2 - \alpha_4)b_6 + (-\alpha_1\alpha_2 - \alpha_2^2 - 2\alpha_2\alpha_4 - \alpha_4^2)b_7}{2}y\\
&- \frac{(\alpha_3b_5 - 2b_8 + \alpha_3b_9)\alpha_3}{2}x + \frac{(b_5 + b_9)}{2}z^3w^2 + \frac{b_7 - b_4}{2}z^2w^3 + \frac{(b_1 + b_3)}{2}z^2w^2\\
&+ \frac{(-3\alpha_1 - 4\alpha_4)b_4 + 2b_6 + (\alpha_1 + 2\alpha_4)b_7}{2}zw^2 + ((\alpha_1 + \alpha_4)b_1 + b_2)zw\\
&+ \frac{(2\alpha_1 + \alpha_4)b_5 + 2b_8 + (-\alpha_4 - 2\alpha_3)b_9}{2}z^2w\\
&+ \frac{\alpha_1(\alpha_1 + \alpha_4)b_5 + 2\alpha_1b_8 + \alpha_1(-\alpha_1 - 2\alpha_3 - \alpha_4)b_9}{2}z\\
&+ \frac{\alpha_2(3\alpha_1 + \alpha_2 + 4\alpha_4)b_4 - 2\alpha_2b_6 + \alpha_2(-\alpha_1 - \alpha_2 - 2\alpha_4)b_7}{2}w + b_9(x^2yzw + \alpha_3xzw)\\
&+ b_1(yz^2w + \alpha_1yz) + b_3xyzw + b_5(xyz^2w + \alpha_1xyz) + 2b_6yzw\\
&+ b_4(xyzw^2 - \frac{5}{2}yz^2w^2 - \frac{3}{2}y^2z^2w - 3\alpha_1yzw - \frac{3}{2}\alpha_1y^2z - \alpha_2(x - z)yw - 2\alpha_4yzw)\\
&+ b_7(xy^2zw + \frac{3}{2}yz^2w^2 + \frac{1}{2}y^2z^2w + \alpha_1yzw + \frac{1}{2}\alpha_1y^2z + \alpha_4yzw).
\end{aligned}
$$

Here, $b_1, b_2, \dots, , b_9$ *are undetermined rational functions in* t.

Propositions 7.1,7.2 and 7.3 can be cheched by a direct calculation, respectively.

8. Proof of Theorems 2.11 and 2.12

As is well-known, the degeneration from P_{VI} to P_V is given by

$$
\begin{gathered}
\alpha_0 = \varepsilon^{-1},\ \alpha_1 = A_3,\ \alpha_3 = A_0 - A_2 - \varepsilon^{-1},\ \alpha_4 = A_1\\
t = 1 + \varepsilon T,\ (x - 1)(X - 1) = 1,\ (x - 1)y + (X - 1)Y = -A_2.
\end{gathered}
$$

Notice that

$$A_0 + A_1 + A_2 + A_3 = \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$$

and the change of variables from (q,p) to (Q,P) is symplectic.

As the fourth-order analogue of the above confluence process, we consider the following coupling confluence process from the system (1). We take the following coupling confluence process $P_{VI} \rightarrow P_V$ for each coordinate system (x,y) and (z,w) of the system (1)

$$\begin{gathered}
\alpha_0 = \varepsilon^{-1}, \quad \alpha_1 = A_0, \quad \alpha_2 = A_1, \quad \alpha_4 - \beta_4 = A_2, \\
\beta_2 = A_3, \quad \beta_3 = -\varepsilon^{-1} - (A_1 + A_2 + A_3 - A_5), \quad \beta_4 = A_4, \\
t = 1 - \varepsilon T, \quad x = \frac{X}{X-T}, \quad z = \frac{Z}{Z-T}, \\
y = -\frac{(X-T)\{(X-T)Y + A_1\}}{T}, \quad w = -\frac{(Z-T)\{(Z-T)W + A_3\}}{T}
\end{gathered}$$

from $\alpha_0, \alpha_1, \alpha_2, \gamma_1, \beta_2, \beta_3, \beta_4, t, x, y, z, w$ to $A_0, \dots, A_5, \varepsilon, T, X, Y, Z, W$. Notice that

$$A_0 + A_1 + A_2 + A_3 + A_4 + A_5 = \alpha_0 + \alpha_1 + 2\alpha_2 + 2(\alpha_4 - \beta_4) + 2\beta_2 + \beta_3 + \beta_4 = 1$$

and the change of variables from (x,y,z,w) to (X,Y,Z,W) is symplectic. Then the system (1) can also be written in the new variables T, X, Y, Z, W and parameters $A_0, A_1, \dots, A_5, \varepsilon$ as a Hamiltonian system. This new system tends to the system (36) of type $A_5^{(1)}$ as $\varepsilon \rightarrow 0$. The proof of Theorem 2.11 is now complete.

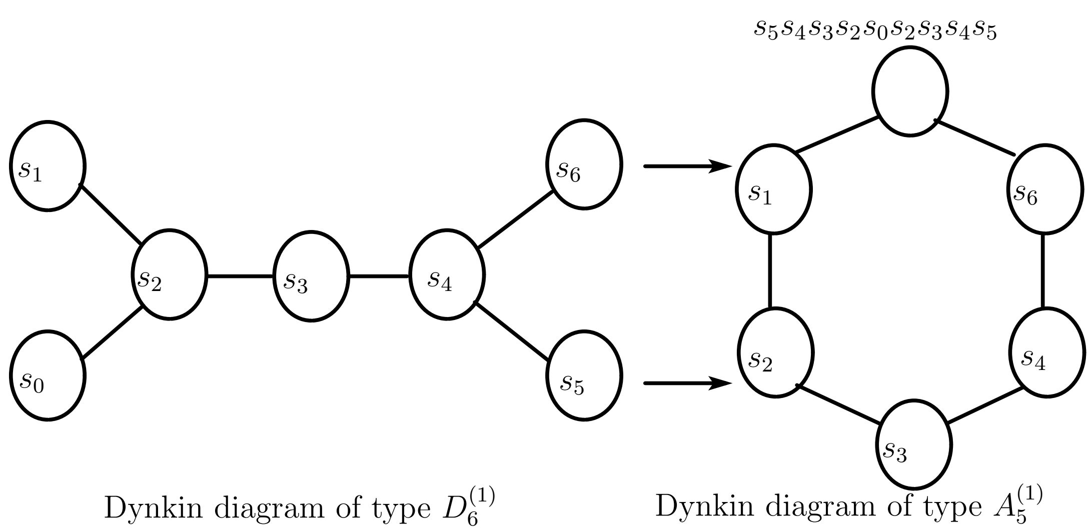

FIGURE 10. Confluence process from the system of $D_6^{(1)}$ to the system of $A_5^{(1)}$

Next, let us prove Theorem 2.12. Choose S_i, $i = 0,1,2,3,4,5$ as

$$S_0 := s_1, \ S_1 := s_2, \ S_2 := s_3, \ S_3 := s_4, \ S_4 := s_6, \ S_5 := s_5s_4s_3s_2s_0s_2s_3s_4s_5,$$

which are reflections of

$$A_0 := \alpha_1, \quad A_1 := \alpha_2, \quad A_2 := \alpha_4 - \beta_4, \quad A_3 := \beta_2,$$
$$A_4 := \beta_4, \quad A_5 := 1 - \alpha_1 - \alpha_2 - \alpha_4 - \beta_2.$$

By using the notation $(*) := (A_0, A_1, A_2, A_3, A_4, A_5, \varepsilon)$, we can easily check

$$\begin{aligned}
S_0(*) &= (-A_0, A_1 + A_0, A_2, A_3, A_4, A_5 + A_0, \varepsilon),\\
S_1(*) &= \left(A_0 + A_1, -A_1, A_2 + A_1, A_3, A_4, A_5, \frac{\varepsilon}{1 + \varepsilon A_1}\right),\\
S_2(*) &= (A_0, A_1 + A_2, -A_2, A_3 + A_2, A_4, A_5, \varepsilon),\\
S_3(*) &= (A_0, A_1, A_2 + A_3, -A_3, A_4 + A_3, A_5, \varepsilon),\\
S_4(*) &= (A_0, A_1, A_2, A_3 + A_4, -A_4, A_5 + A_4, \varepsilon),\\
S_5(*) &= \left(A_0 + A_5, A_1, A_2, A_3, A_4 + A_5, -A_5, \frac{\varepsilon}{1 - \varepsilon A_5}\right).
\end{aligned}$$

By the above relation, we will see that the group $< S_0, S_1, \dots, S_5 >$ can be considered to be an affine Weyl group of the affine Lie algebra of type $A_5^{(1)}$ with respect to simple roots $A_0, A_1, \dots, A_5$.

Now we investigate how the generators of $< S_0, S_1, \dots, S_5 >$ act on X, Y, Z, W. By using the notation $(**) := (X, Y, Z, W)$, we can verify

$$\begin{aligned}
S_0(**) &= \left(X, Y - \frac{A_0}{X - T}, Z, W\right),\\
S_1(**) &= \left(X + \frac{A_1}{Y}, Y, Z, W\right),\\
S_2(**) &= \left(X, Y - \frac{A_2}{X - Z}, Z, W + \frac{A_2}{X - Z}\right),\\
S_3(**) &= \left(X, Y, Z + \frac{A_3}{W}, W\right),\\
S_4(**) &= \left(X, Y, Z, W - \frac{A_4}{Z}\right),\\
S_5(**) &= \left(X + \frac{A_5}{Y + W - 1}, Y, Z + \frac{A_5}{Y + W - 1}, W\right).
\end{aligned}$$

The proof of Theorem 2.12 has thus been completed. □

9. Appendix

For the second-order Painlevé equations, we can obtain the entire space of initial conditions by adding subvarieties of codimension 1 (equivalently, of dimension 1) to the space of initial conditions of holomorphic solutions. However, in the case of the fourth-order differential equations, we need to add codimension 2 subvarieties to the space in addition to codimension 1 subvarieties (see [16, 17, 18]). In order to resolve singularities, we need

to both blow up and blow down. Moreover, to obtain a smooth variety by blowing-down, we need to resolve for a pair of singularities (see [16, 17, 18]). In this section, we will give some canonical coordinate systems of the system (1). Each of them contains a 3-parameter or 2-parameter family of meromorphic solutions.

In order to consider the singularity analysis for the system (1), let us take the compactification $([z_0 : z_1 : z_2 : z_3 : z_4], t) \in \mathbb{P}^4 \times B$ of $(x, y, z, w, t) \in \mathbb{C}^4 \times B$, $B := \mathbb{C} - \{0, 1\}$ with the natural embedding

$$(x, y, z, w) = (z_1/z_0, z_2/z_0, z_3/z_0, z_4/z_0).$$

Moreover, we denote the boundary divisor in $\mathbb{P}^4$ by $\mathcal{H}$. Fixing the parameter $\alpha_0, \alpha_1, \beta_0$, consider the product $\mathbb{P}^4 \times B$ and extend the regular vector field on $\mathbb{C}^4 \times B$ to a rational vector field $\tilde{v}$ on $\mathbb{P}^4 \times B$. It is easy to see that $\mathbb{P}^4 \times B$ is covered by five copies of $\mathbb{C}^4 \times B$:

$$\begin{gathered} U_0 \times B = \mathbb{C}^4 \times B \ni (x, y, z, w, t), \\ U_j \times B = \mathbb{C}^4 \times B \ni (X_j, Y_j, Z_j, W_j, t) \ (j = 1, 2, 3, 4) \end{gathered}$$

via the following rational transformations

$$\begin{aligned} &1) X_1 = 1/x, \quad Y_1 = y/x, \quad Z_1 = z/x, \quad W_1 = w/x, \\ &2) X_2 = x/z, \quad Y_2 = y/z, \quad Z_2 = 1/z, \quad W_2 = w/z, \\ &3) X_3 = x/y, \quad Y_3 = 1/y, \quad Z_3 = z/y, \quad W_3 = w/y, \\ &4) X_4 = x/w, \quad Y_4 = y/w, \quad Z_4 = z/w, \quad W_4 = 1/w. \end{aligned}$$

The following lemma shows that this rational vector field $\tilde{v}$ has the following five accessible singular loci on the boundary divisor $\mathcal{H} \times \{\mathrm{t}\} \subset \mathbb{P}^4 \times \{\mathrm{t}\}$ for each $t \in B$.

Lemma 9.1. *The rational vector field $\tilde{v}$ has the following accessible singular loci*:

$$\begin{cases} P_i = \{(X_i, Y_i, Z_i, W_i) | X_i = Y_i = Z_i = W_i = 0\} \ (1 = 1, 2, 3, 4), \\ P_5 = \{(X_3, Y_3, Z_3, W_3) | X_3 = Y_3 = Z_3 = 0, W_3 = -1\}. \end{cases}$$

This lemma can be proven by a direct calculation. □

Proposition 9.2. *If we resolve the accessible singular points given in Lemma 9.1 by blowing-ups, then we can obtain the canonical coordinates $r_j (j = 0, 1, \dots, 6)$ given in Theorem 2.10.*

Proof By the following steps, we can resolve the accessible singular point P_1.

Step 1: We blow up at the point P_1:

$$X_1{}^{(1)} = X_1 \ , \quad Y_1{}^{(1)} = \frac{Y_1}{X_1} \ , \quad Z_1{}^{(1)} = \frac{Z_1}{X_1} \ , \quad W_1{}^{(1)} = \frac{W_1}{X_1}.$$

Step 2: We blow up along the surface $\{({X_1}^{(1)},{Y_1}^{(1)},{Z_1}^{(1)},{W_1}^{(1)})|{X_1}^{(1)}={Y_1}^{(1)} =0\}$:

$${X_1}^{(2)}={X_1}^{(1)} , \quad {Y_1}^{(2)}=\frac{{Y_1}^{(1)}}{{X_1}^{(1)}} , \quad {Z_1}^{(2)}={Z_1}^{(1)} , \quad {W_1}^{(2)}={W_1}^{(1)}.$$

It is easy to see that there are two accessible singular loci:

$$S_1^{(1)}=\{({X_1}^{(2)},{Y_1}^{(2)},{Z_1}^{(2)},{W_1}^{(2)})|{X_1}^{(2)}={Y_1}^{(2)}+\alpha_1+\alpha_2=0\},$$
$$S_1^{(2)}=\{({X_1}^{(2)},{Y_1}^{(2)},{Z_1}^{(2)},{W_1}^{(2)})|{X_1}^{(2)}={Y_1}^{(2)}+\alpha_2=0\}.$$

Step 3: We blow up along the surface $S_1^{(1)}$:

$${X_1}^{(3)}={X_1}^{(2)} , \quad {Y_1}^{(3)}=\frac{{Y_1}^{(2)}+\alpha_1+\alpha_2}{{X_1}^{(2)}} , \quad {Z_1}^{(3)}={Z_1}^{(2)} , \quad {W_1}^{(3)}={W_1}^{(2)}.$$

Step 4: We blow up along the surface $S_1^{(2)}$:

$${X_1}^{(4)}={X_1}^{(2)} , \quad {Y_1}^{(4)}=\frac{{Y_1}^{(2)}+\alpha_2}{{X_1}^{(2)}} , \quad {Z_1}^{(4)}={Z_1}^{(2)} , \quad {W_1}^{(4)}={W_1}^{(2)}.$$

Thus we have resolved the accessible singular point P_1.

By choosing new coordinate systems as

$$(x_k,y_k,z_k,w_k)=({X_1}^{(k+2)},-{Y_1}^{(k+2)},{Z_1}^{(k+2)},{W_1}^{(k+2)}) \ (k=1,2),$$

we can obtain the coordinate systems (x_k,y_k,z_k,w_k) $(k=1,2)$, respectively.

By the following steps, we can resolve the accessible singular point P_4.

Step 1: We blow up at the point P_4:

$${X_4}^{(1)}=\frac{X_4}{W_4} , \quad {Y_4}^{(1)}=\frac{Y_4}{W_4} , \quad {Z_4}^{(1)}=\frac{Z_4}{W_4} , \quad {W_4}^{(1)}=W_4.$$

It is easy to see that there are two accessible singular loci:

$$S_4^{(1)}=\{({X_4}^{(1)},{Y_4}^{(1)},{Z_4}^{(1)},{W_4}^{(1)})|{Z_4}^{(1)}-1={W_4}^{(1)}=0\},$$
$$S_4^{(2)}=\{({X_4}^{(1)},{Y_4}^{(1)},{Z_4}^{(1)},{W_4}^{(1)})|{Z_4}^{(1)}={W_4}^{(1)}=0\}.$$

Step 2: We blow up along the surface $S_4^{(1)}$:

$${X_4}^{(2)}={X_4}^{(1)} , \quad {Y_4}^{(2)}={Y_4}^{(1)} , \quad {Z_4}^{(2)}=\frac{{Z_4}^{(1)}-1}{{W_4}^{(1)}} , \quad {W_4}^{(2)}={W_4}^{(1)}.$$

Step 3: We blow up along the surface $\{({X_4}^{(2)},{Y_4}^{(2)},{Z_4}^{(2)},{W_4}^{(2)})|{Z_4}^{(2)}-\beta_3 ={W_4}^{(2)}=0\}$:

$${X_4}^{(3)}={X_4}^{(2)} , \quad {Y_4}^{(3)}={Y_4}^{(2)} , \quad {Z_4}^{(3)}=\frac{{Z_4}^{(2)}-\beta_3}{{W_4}^{(2)}} , \quad {W_4}^{(3)}={W_4}^{(2)}.$$

Step 4: We blow up along the surface $S_4^{(2)}$:

$$X_4{}^{(4)}=X_4{}^{(1)}\ ,\quad Y_4{}^{(4)}=Y_4{}^{(1)}\ ,\quad Z_4{}^{(4)}=\frac{Z_4{}^{(1)}}{W_4{}^{(1)}}\ ,\quad W_4{}^{(4)}=W_4{}^{(1)}.$$

Step 5: We blow up along the surface $\{(X_4{}^{(4)},Y_4{}^{(4)},Z_4{}^{(4)},W_4{}^{(4)})|Z_4{}^{(4)}-\beta_4 = W_4{}^{(4)}=0\}$:

$$X_4{}^{(5)}=X_4{}^{(4)}\ ,\quad Y_4{}^{(5)}=Y_4{}^{(4)}\ ,\quad Z_4{}^{(5)}=\frac{Z_4{}^{(4)}-\beta_4}{W_4{}^{(4)}}\ ,\quad W_4{}^{(5)}=W_4{}^{(4)}.$$

Thus we have resolved the accessible singular point P_4.

By choosing new coordinate systems as

$$(x_5,y_5,z_5,w_5)=(X_4{}^{(3)},Y_4{}^{(3)},-Z_4{}^{(3)},W_4{}^{(3)})$$
$$(x_6,y_6,z_6,w_6)=(X_4{}^{(5)},Y_4{}^{(5)},-Z_4{}^{(5)},W_4{}^{(5)}),$$

we can obtain the coordinate systems (x_k,y_k,z_k,w_k) $(k=5,6)$, respectively.

By the following steps, we can resolve the accessible singular point P_5.

Step 0: We take the coordinate system centered at P_5:

$$X_5{}^{(0)}=X_3\ ,\quad Y_5{}^{(0)}=Y_3\ ,\quad Z_5{}^{(0)}=Z_3\ ,\quad W_5{}^{(0)}=W_3+1.$$

Step 1: We blow up at the point P_5:

$$X_5{}^{(1)}=\frac{X_5{}^{(0)}}{Y_5{}^{(0)}}\ ,\quad Y_5{}^{(1)}=Y_5{}^{(0)}\ ,\quad Z_5{}^{(1)}=\frac{Z_5{}^{(0)}}{Y_5{}^{(0)}}\ ,\quad W_5{}^{(1)}=\frac{W_5{}^{(0)}}{Y_5{}^{(0)}}.$$

Step 2: We blow up along the surface $\{(X_5{}^{(1)},Y_5{}^{(1)},Z_5{}^{(1)},W_5{}^{(1)})|X_5{}^{(1)}-Z_5{}^{(1)} = Y_5{}^{(1)}=0\}$:

$$X_5{}^{(2)}=\frac{X_5{}^{(1)}-Z_5{}^{(1)}}{Y_5{}^{(1)}}\ ,\quad Y_5{}^{(2)}=Y_5{}^{(1)}\ ,\quad Z_5{}^{(2)}=Z_5{}^{(1)}\ ,\quad W_5{}^{(2)}=W_5{}^{(1)}.$$

Step 3: We blow up along the surface $\{(X_5{}^{(2)},Y_5{}^{(2)},Z_5{}^{(2)},W_5{}^{(2)})|X_5{}^{(2)}- (\alpha_4-\beta_4)=Y_5{}^{(2)}=0\}$:

$$X_5{}^{(3)}=\frac{X_5{}^{(2)}-(\alpha_4-\beta_4)}{Y_5{}^{(2)}}\ ,\quad Y_5{}^{(3)}=Y_5{}^{(2)}\ ,\quad Z_5{}^{(3)}=Z_5{}^{(2)}\ ,\quad W_5{}^{(3)}=W_5{}^{(2)}.$$

Thus we have resolved the accessible singular point P_5.

By choosing a new coordinate system as

$$(x_3,y_3,z_3,w_3)=(-X_5{}^{(3)},Y_5{}^{(3)},Z_5{}^{(3)},W_5{}^{(3)}),$$

we can obtain the coordinate system (x_3,y_3,z_3,w_3).

For the remaining accessible singular points, the proof is similar.

Collecting all the cases, we have obtained the canonical coordinate systems (x_j,y_j, z_j,w_j) $(j=0,1,\dots,6)$, which proves Proposition 10.1. $\square$

We remark that each coordinate system contains a three-parameter family of meromorphic solutions of (1) as the initial conditions.

By using the coordinate systems (x_j, y_j, z_j, w_j) $(j = 0, 1, 2, \dots, 6)$, we will now make coordinate systems associated with other small meromorphic solution spaces. For example, we can take the coordinate system $(x_3, y_3, z_3, w_3) = (-((x-z)y-(\alpha_4-\beta_4))y, 1/y, z, y+w)$. As a boundary coordinate system of this system (x_3, y_3, z_3, w_3), we can take the coordinate system

$$(X_3^{(0)}, Y_3^{(0)}, Z_3^{(0)}, W_3^{(0)}) = (x_3, y_3, z_3, 1/w_3).$$

It is easy to see that there is an accessible singularity along the surface

$$S_3 = \{(X_3^{(0)}, Y_3^{(0)}, Z_3^{(0)}, W_3^{(0)}) | Z_3^{(0)} = W_3^{(0)} = 0\}.$$

Now we blow up along the accessible singularity S_3.

Step 1: We blow up along the surface S_3:

$$X_3{}^{(1)} = X_3^{(0)}, \quad Y_3{}^{(1)} = Y_3^{(0)}, \quad Z_3{}^{(1)} = \frac{Z_3^{(0)}}{W_3^{(0)}}, \quad W_3{}^{(1)} = W_3^{(0)}.$$

Step 2: We blow up along the surface $\{(X_3^{(1)}, Y_3^{(1)}, Z_3^{(1)}, W_3^{(1)}) | Z_3^{(1)} - \beta_4 = W_3^{(1)} = 0\}$:

$$X_3{}^{(2)} = X_3{}^{(1)}, \quad Y_3{}^{(2)} = Y_3{}^{(1)}, \quad Z_3{}^{(2)} = \frac{Z_3{}^{(1)} - \beta_4}{W_3{}^{(1)}}, \quad W_3{}^{(2)} = W_3{}^{(1)}.$$

Thus we have resolved the accessible singularity S_3. By the same way, we can obtain the following canonical coordinate systems.

Proposition 9.3. *The system* (1) *has the following canonical coordinate systems with regard to the transformations* $r_i r_j$*:*

$$\begin{aligned}
r_0r_3 :& x_7 = -(y+w)((x-t)(y+w)-\alpha_0), \quad y_7 = 1/(y+w),\\
& z_7 = -((z-x)w-(\alpha_4-\beta_4))w, \quad w_7 = 1/w,\\
r_0r_3 :& x_7 = -(y+w)((x-t)(y+w)-\alpha_0), \quad y_7 = 1/(y+w),\\
& z_7 = -((z-x)w-(\alpha_4-\beta_4))w, \quad w_7 = 1/w,\\
r_0r_4 :& x_8 = -((x-t)y-\alpha_0)y, \quad y_8 = 1/y,\\
& z_8 = 1/z, \quad w_8 = -z(zw+\beta_2),\\
r_0r_5 :& x_9 = -((x-t)y-\alpha_0)y, \quad y_9 = 1/y,\\
& z_9 = -((z-1)w-\beta_3)w, \quad w_9 = 1/w,\\
r_0r_6 :& x_{10} = -((x-t)y-\alpha_0)y, \quad y_{10} = 1/y,\\
& z_{10} = -(zw-\beta_4)w, \quad w_{10} = 1/w,\\
r_1r_4 :& x_{11} = 1/x, \quad y_{11} = -(xy+\alpha_1+\alpha_2)x,\\
& z_{11} = 1/z, \quad w_{11} = -z(zw+\beta_2),\\
r_1r_5 :& x_{12} = 1/x, \quad y_{12} = -(xy+\alpha_1+\alpha_2)x,\\
& z_{12} = -((z-1)w-\beta_3)w, \quad w_{12} = 1/w,
\end{aligned}$$

$$\begin{aligned}
r_1r_6 :& x_{13} = 1/x, \quad y_{13} = -(xy+\alpha_1+\alpha_2)x,\\
& z_{13} = -(zw-\beta_4)w, \quad w_{13} = 1/w,\\
r_2r_4 :& x_{14} = 1/x, \quad y_{14} = -x(xy+\alpha_2),\\
& z_{14} = 1/z, \quad w_{14} = -z(zw+\beta_2),\\
r_2r_5 :& x_{15} = 1/x, \quad y_{15} = -x(xy+\alpha_2),\\
& z_{15} = -((z-1)w-\beta_3)w, \quad w_{15} = 1/w,\\
r_2r_6 :& x_{16} = 1/x, \quad y_{16} = -x(xy+\alpha_2),\\
& z_{16} = -(zw-\beta_4)w, \quad w_{16} = 1/w,\\
r_3r_5 :& x_{17} = -((x-z)y-(\alpha_4-\beta_4))y, \quad y_{17} = 1/y,\\
& z_{17} = -((z-1)(y+w)-\beta_3)(y+w), \quad w_{17} = 1/(y+w),\\
r_3r_6 :& x_{18} = -((x-z)y-(\alpha_4-\beta_4))y, \quad y_{18} = 1/y,\\
& z_{18} = -(z(y+w)-\beta_4)(y+w), \quad w_{18} = 1/(y+w).
\end{aligned}$$

Each coordinate system contains a two-parameter family of meromorphic solutions of (1) as the initial conditions. By using the coordinate systems (x_j, y_j, z_j, w_j) $(j = 7, 8, \dots, 18)$, we will now make the coordinate systems associated with other small meromorphic solution spaces by the same way. For example, we can take the coordinate system $(x_{15}, y_{15}, z_{15}, w_{15}) = (-((x-z)y-(\alpha_4-\beta_4))y, 1/y, -(z(y+w)-\beta_4)(y+w), 1/(y+w))$. As a

boundary coordinate system of this system $(x_{15}, y_{15}, z_{15}, w_{15})$, we can take the coordinate system

$$(X_{15}, Y_{15}, Z_{15}, W_{15}) = (x_{15} + z_{15}, y_{15}, 1/z_{15}, w_{15} - y_{15}).$$

It is easy to see that there is an accessible singularity along the surface

$$S_{15} = \{(X_{15}, Y_{15}, Z_{15}, W_{15}) | Z_{15} = W_{15} = 0\}.$$

Now we blow up along the accessible singularity S_{15}.

Step 1: We blow up along the surface $\{(X_{15}, Y_{15}, Z_{15}, W_{15}) | Z_{15} = W_{15} = 0\}$:

$${X_{15}}^{(1)} = X_{15} \ , \quad {Y_{15}}^{(1)} = Y_{15} \ , \quad {Z_{15}}^{(1)} = Z_{15} \ , \quad {W_{15}}^{(1)} = \frac{W_{15}}{Z_{15}}.$$

Step 2: We blow up along the surface $\{({X_{15}}^{(1)}, {Y_{15}}^{(1)}, {Z_{15}}^{(1)}, {W_{15}}^{(1)}) | {Z_{15}}^{(1)} = {W_{15}}^{(1)} + \beta_2 = 0\}$:

$${X_{15}}^{(2)} = {X_{15}}^{(1)} \ , \quad {Y_{15}}^{(2)} = {Y_{15}}^{(1)} \ , \quad {Z_{15}}^{(2)} = {Z_{15}}^{(1)} \ , \quad {W_{15}}^{(2)} = \frac{{W_{15}}^{(1)} + \beta_2}{{Z_{15}}^{(1)}}.$$

Thus we have resolved the accessible singularity S_{15}. By the same way, we can obtain the following canonical coordinate systems.

PROPOSITION 9.4. *The system* (1) *has the following canonical coordinate systems with regard to the transformations* $r_i r_j r_k$:

$r_3(r_4r_2)$:
$x_{19} = 1/x, \quad y_{19} = -x^2y - z^2w - \alpha_2 x - \beta_2 z,$
$z_{19} = z(zw + \beta_2)(-xzw + z^2w - \alpha_4 x - \beta_2 x + \beta_2 z + \beta_4 x)/x,$
$w_{19} = -1/(z(zw + \beta_2)),$
$r_4(r_5r_3)$:
$x_{20} = w^2 + 2yw + y^2 - xy^2 - zw^2 - 2yzw + \alpha_4 y + \beta_3(y + w) - \beta_4 y,$
$y_{20} = 1/y, \quad z_{20} = -1/((y + w)(-y - w + zw + yz - \beta_3)),$
$w_{20} = -((y + w)(-y - w + zw + yz - \beta_3)((y + w)(zw - w) + \beta_2 y - \beta_3 w))/y,$
$r_4(r_6r_3)$:
$x_{21} = -xy^2 - zw^2 - 2yzw + \alpha_4 y + \beta_4 w, \quad y_{21} = 1/y,$
$z_{21} = -1/((y + w)(zw + yz - \beta_4)),$
$w_{21} = -(y + w)(zw + yz - \beta_4)(zw^2 + yzw + \beta_2 y - \beta_4 w)/y,$
$r_2(r_0r_3)$:
$x_{22} = 1/((y + w)(tw - xw + ty - xy + \alpha_0)),$
$y_{22} = (y + w)(tw - xw + ty - xy + \alpha_0)(tyw - xyw + ty^2 - xy^2 + \alpha_0 y - \alpha_2 w)/w,$
$z_{22} = tw^2 + 2(t - x)yw + ty^2 - xy^2 - zw^2 + \alpha_0(y + w) + (\alpha_4 - \beta_4)w, \ w_{22} = 1/w,$
$r_3(r_4r_1)$:
$x_{23} = 1/x, \quad y_{23} = -x^2y - z^2w - (\alpha_1 + \alpha_2)x - \beta_2 z,$
$z_{23} = z(zw + \beta_2)(-xzw + z^2w - \alpha_4 x - \beta_2 x + \beta_2 z + \beta_4 x)/x,$
$w_{23} = -1/(z(zw + \beta_2)),$
$r_1(r_0r_3)$:
$x_{24} = 1/((y + w)(tw - xw + ty - xy + \alpha_0)),$
$y_{24} = (y + w)((t - x)(y + w) + \alpha_0)(tyw - xyw + ty^2 - xy^2 + \alpha_0 y - (\alpha_1 + \alpha_2)w)/w,$
$z_{24} = tw^2 + 2(t - x)yw + ty^2 - xy^2 - zw^2 + \alpha_0(y + w) + (\alpha_4 - \beta_4)w, \quad w_{24} = 1/w.$

Each of them contains a two-parameter family of meromorphic solutions of (1) as the initial conditions.

It is still an open question whether the phase space of the system (1) can be covered by the original coordinate system (x, y, z, w) in addition to the canonical coordinate systems (x_i, y_i, z_i, w_i) $(i = 0, 1, \dots, 24)$.

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